

ON ENDOSCOPIC DECOMPOSITION OF CERTAIN DEPTH ZERO REPRESENTATIONS

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Dedicated to A. Joseph on his 60th birthday

ABSTRACT. We construct an endoscopic decomposition for local L -packets associated to irreducible cuspidal Deligne–Lusztig representations. Moreover, the obtained decomposition is compatible with inner twistings.

Introduction

Let E be a local non-archimedean field, $\Gamma \supset W \supset I$ the absolute Galois, the Weil and the inertia groups of E , respectively. Let G be a reductive group over E , and let ${}^L G = \widehat{G} \rtimes W$ be the complex Langlands dual group of G . Denote by $\mathcal{D}(G(E))$ the space of invariant generalized functions on $G(E)$, that is, the space of $\text{Int } G$ -invariant linear functionals on the space of locally constant compactly supported measures on $G(E)$.

Every admissible homomorphism $\lambda : W \rightarrow {}^L G$ (see [Ko1, § 10]) gives rise to a finite group $S_\lambda := \pi_0(Z_{\widehat{G}}(\lambda)/Z(\widehat{G})^\Gamma)$, where $Z_{\widehat{G}}(\lambda)$ is the centralizer of $\lambda(W)$ in \widehat{G} . To every conjugacy class κ of S_λ , Langlands [La1] associated an endoscopic subspace $\mathcal{D}_{\kappa,\lambda}(G(E)) \subset \mathcal{D}(G(E))$. For simplicity, we will restrict ourselves to the elliptic case, where $\lambda(W)$ does not lie in any proper Levi subgroup of ${}^L G$.

Langlands conjectured that every elliptic λ corresponds to a finite set Π_λ , called an L -packet, of cuspidal irreducible representations of $G(E)$. Moreover, the subspace $\mathcal{D}_\lambda(G(E)) \subset \mathcal{D}(G(E))$, generated by characters $\{\chi(\pi)\}_{\pi \in \Pi_\lambda}$, should have an endoscopic decomposition. More precisely, it is expected ([La1, IV, 2]) that there exists a basis $\{a_\pi\}_{\pi \in \Pi_\lambda}$ of the space of central functions on S_λ such that $\chi_{\kappa,\lambda} := \sum_{\pi \in \Pi_\lambda} a_\pi(\kappa) \chi(\pi)$ belongs to $\mathcal{D}_{\kappa,\lambda}(G)$ for every conjugacy class κ of S_λ .

The goal of this paper is to construct the endoscopic decomposition of $\mathcal{D}_\lambda(G(E))$ for tamely ramified λ 's such that $Z_{\widehat{G}}(\lambda(I))$ is a maximal torus. In this case, G splits

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over an unramified extension of E , and λ factors through ${}^L T \hookrightarrow {}^L G$ for an elliptic unramified maximal torus T of G .

Each $\kappa \in S_\lambda = \widehat{T}^\Gamma / Z(\widehat{G})^\Gamma$ gives rise to an elliptic endoscopic triple $\mathcal{E}_{\kappa, \lambda}$ for G , while characters of S_λ are in bijection with conjugacy classes of embeddings $T \hookrightarrow G$, stably conjugate to the inclusion. By the local Langlands correspondence for tori ([La2]), a homomorphism $\lambda : W \rightarrow {}^L T$ defines a tamely ramified homomorphism $\theta : T(E) \rightarrow \mathbb{C}^\times$. Therefore each character a of S_λ gives rise to an irreducible cuspidal representation $\pi_{a, \lambda}$ of $G(E)$ (denoted by $\pi_{a, \theta}$ in Notation 2.1.3).

Our main theorem asserts that if the residual characteristic of E is sufficiently large, then each $\chi_{\kappa, \lambda} := \sum_a a(\kappa) \chi(\pi_{a, \lambda})$ is $\mathcal{E}_{\kappa, \lambda}$ -stable (see Definition 1.6.6). More generally (see Corollary 1.6.12 (b)), for each inner form G' of G , we denote by $\chi'_{\kappa, \lambda}$ the corresponding generalized function on $G'(E)$, and our main theorem asserts that $\chi_{\kappa, \lambda}$ and $\chi'_{\kappa, \lambda}$ are “ $\mathcal{E}_{\kappa, \lambda}$ -equivalent” (see Definition 1.6.10 for a more precise term). Though in this work we show this result only for local fields of characteristic zero, the case of local fields of positive characteristic follows by approximation (see [KV2]).

Our argument goes as follows. First we prove the equivalence of the restrictions of $\chi_{\kappa, \lambda}$ and $\chi'_{\kappa, \lambda}$ to the subsets of topologically unipotent elements of $G(E)$ and $G'(E)$. If the residual characteristic of E is sufficiently large, topologically unipotent elements of $G(E)$ and $G'(E)$ can be identified with topologically nilpotent elements of the Lie algebras $\mathcal{G}(E)$ and $\mathcal{G}'(E)$, respectively. Thus we are reduced to an analogous assertion about generalized functions on Lie algebras. Now the equivalence follows from a combination of a Springer hypothesis (Theorem A.1), which describes the trace of a Deligne–Lusztig representation in terms of Fourier transform of an orbit, and a generalization a theorem of Waldspurger [Wa2] to inner forms, which asserts that up to a sign, Fourier transform preserves the equivalence.

To prove the result in general, we use the topological Jordan decomposition ([Ka2]). We would like to stress that in order to prove just the stability of $\chi_{\kappa, \lambda}$ one still needs a generalization of [Wa2] to inner forms.

This paper is organized as follows.

In the first section we give basic definitions and constructions of a rather general nature. In particular, most of the section is essentially a theory of endoscopy, which was developed by Langlands, Shelstad and Kottwitz. In order to incorporate both the case of algebraic groups and of Lie algebras we work in a more general context of algebraic varieties equipped with an action of G^{ad} .

More precisely, in Subsection 1.1 we recall basic properties, results and constructions concerning inner twistings and stable conjugacy. In Subsections 1.2 and 1.3 we give basic definitions and properties of dual groups and of endoscopic triples. Then in Subsection 1.4 we prove that certain subsets of the group $Z(\widehat{G^{\text{ad}}})^\Gamma$ are

actually subgroups. Unfortunately, this result is proven case-by-case. In Subsection 1.5 we specialize previous results to the case of endoscopic triples over local non-archimedean fields.

In Subsection 1.6 we define the notions of stability and equivalence of generalized functions, while in Subsection 1.7 we write down explicitly the condition for stability and equivalence for generalized functions coming from invariant locally L^1 functions. Note that the notion of equivalence is much more subtle than that of stability. In particular, it depends not just on an endoscopic triple but also on a triple $(a, a'; [b])$, consisting of compatible embeddings of maximal tori into G , G' and the endoscopic group.

We finish the section by Subsection 1.8 in which we study basic properties of certain equivariant maps from reductive groups to their Lie algebras, which we call quasi-logarithms. We use these maps to identify topologically unipotent elements of the group with topologically nilpotent elements of the Lie algebra.

The second section is devoted to the formulation and the proof of the main theorem. More precisely, in Subsection 2.1 we give two equivalent formulations of our main result. In Subsection 2.2 we prove the equivalence of the restrictions of $\chi_{\kappa, \lambda}$ and $\chi'_{\kappa, \lambda}$ to topologically unipotent elements.

In Subsection 2.3 we rewrite character $\chi(\pi_{a, \theta})$ of $G(E)$ in terms of restrictions to topologically unipotent elements of corresponding characters of centralizers $G_\delta(E)$. For this we use the topological Jordan decomposition. In the next Subsection 2.4 we compare endoscopic triples for the group G and for its centralizers $G_\delta(E)$. Finally, in Subsections 2.5–2.7 we carry out the proof itself.

We finish the paper by two appendices of independent interest, crucially used in Subsection 2.2. In Appendix A we prove Springer hypothesis. In the case of large characteristic this result was proved by the first author in [Ka1]. For the proof in general, we use Lusztig's interpretation of a trace of Deligne–Lusztig representation in terms of character sheaves [Lu] and results of Springer [Sp] on Fourier transform.

In Appendix B we prove a generalization of both the theorem of Waldspurger [Wa2] and that of Kazhdan–Polishchuk [KP, Thm. 2.7.1] (see also Remark B.1.3). Our strategy is very similar to those of [Wa2] and [KP]. More precisely, using stationary phase principle and the results of Weil [We], we construct in Subsection B.2 certain measures whose Fourier transform can be explicitly calculated. Then in Subsection B.3 we extend our data over a local field to a corresponding data over a number field. Finally, in Section 1.4 we deduce our result from a simple form of the trace formula.

For the convenience of the reader, we also include a list of main terms and symbols, indexed by page number they first appear.

This work is an expanded version of the announcement [KV1]. In the process of writing, we have learned that DeBacker and Reeder obtained similar results (see [DBR]). After the work was completed, it was pointed out to us that our scheme of the argument is similar to the one used by Mœglin-Waldspurger in [MW].

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Notation and Conventions

For a finite abelian group A , we denote by A^D the group of complex characters of A .

For an algebraic group G , we denote by G^0 , $Z(G)$, G^{ad} , G^{der} and $W(G)$ the connected component of the identity of G , the center of G , the adjoint group of G , the derived group of G , and the Weyl group of G , respectively. Starting from 1.1.9, G will be always assumed to be reductive and connected, in which case we denote by G^{sc} the simply connected covering of G^{der} .

We denote by \mathcal{G} , \mathcal{H} , \mathcal{T} and \mathcal{L} Lie algebras of algebraic groups G , H , T and L , respectively.

Let an algebraic group G acts on an algebraic variety X . For each $x \in X$, we denote by G_x and \mathcal{G}_x the stabilizers of x in G and \mathcal{G} , respectively. Explicitly, \mathcal{G}_x is the kernel of the differential at $g = 1$ of the morphism $G \rightarrow X$ ($g \mapsto g(x)$). (Note that that x and therefore also G_x and \mathcal{G}_x will have different meaning starting from 1.8.5.)

Each algebraic group acts on itself by inner automorphisms and by adjoint action on its Lie algebra. For each $g \in G$ we denote by $\text{Int } g$ and $\text{Ad } g$ the corresponding elements in $\text{Int } G \subset \text{Aut}(G)$ and $\text{Ad } G \subset \text{Aut}(\mathcal{G})$, respectively.

For a field E , we denote by \overline{E} a fixed algebraic closure of E , and by E^{sep} the maximal separable extension of E in \overline{E} . Γ will be always the absolute Galois group of E . When Γ acts on a set X we will write ${}^\sigma x$ instead of $\sigma(x)$.

For a reductive algebraic group G over E , we denote by $\text{rk}_E(G)$ the rank of G over E , and put $e(G) := (-1)^{\text{rk}_E(G^{\text{ad}})}$. We also set $e'(G) := e(G)e(G^*)$, where G^* is the quasi-split inner form of G . Then $e'(G)$ coincides with the sign defined by Kottwitz ([Ko5]).

Starting from Subsection 1.5, E will be a local non-archimedean field with ring of integers \mathcal{O} , maximal ideal \mathfrak{m} , and residue field \mathbb{F}_q of characteristic p . We denote by E^{nr} the maximal unramified extension of E in \overline{E} . Starting from Subsection 2.2, we will assume that the characteristic of E is zero.

CONTENTS

1. Basic definitions and constructions	5
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1.1. Stable conjugacy	5
1.2. Preliminaries on dual groups	10
1.3. Endoscopic triples: basic properties	13
1.4. Endoscopic triples: further properties.	18
1.5. The case of local fields	23
1.6. Definitions of stability and equivalence	26
1.7. Locally L^1 functions	31
1.8. Quasi-logarithm maps	36
2. Endoscopic decomposition	42
2.1. Main Theorem	42
2.2. Stability of the restriction to $G(E)_{\text{tu}}$	45
2.3. Reduction formula	50
2.4. Endoscopy for G and G_δ	55
2.5. Preparation for the proof of the Main Theorem	59
2.6. Proof of Theorem 2.1.10 (i)	63
2.7. Proof of Theorem 2.1.10 (ii)	65
Appendix A. Springer Hypothesis	66
Appendix B. $(a, a'; [b])$ -equivalence and Fourier transform	69
B.1. Formulation of the result	69
B.2. Local calculations	71
B.3. Global results	75
B.4. Proof of Theorem B.1.4	79
List of main terms and symbols	83
References	84

1. BASIC DEFINITIONS AND CONSTRUCTIONS

1.1. Stable conjugacy.

In this subsection we recall basic definitions and constructions concerning inner forms and stable conjugacy.

Let G be an algebraic group over a field E . Starting from 1.1.9, we will assume that G is reductive and connected.

Let X be an algebraic variety over E (that is, a reduced scheme locally of finite type over E) equipped with an action of G^{ad} (that is, with an action of G , trivial on $Z(G)$).

Our basic examples will be $X = G$ and $X = \mathcal{G}$ with the natural action of G^{ad} .

1.1.1. Inner twistings. (a) Let G' be an algebraic group over E . Recall that an *inner twisting* $\varphi : G \rightarrow G'$ is an isomorphism $\varphi : G_{E^{\text{sep}}} \xrightarrow{\sim} G'_{E^{\text{sep}}}$ such that for each $\sigma \in \Gamma$ the automorphism $c_\sigma := \varphi^{-1}\sigma\varphi \in \text{Aut}(G)$ is inner. In this case, $\{c_\sigma\}_\sigma$ form a cocycle of Γ in $\text{Int } G = G^{\text{ad}}$, and we denote by $\text{inv}(\varphi) = \text{inv}(G, G') \in H^1(E, G^{\text{ad}})$ the corresponding cohomology class.

(b) Two inner twistings are called *isomorphic* if they differ by an inner automorphism. Then the map $(\varphi : G \rightarrow G') \mapsto \text{inv}(G, G')$ gives a bijection between the set of isomorphism classes of inner twistings of G and $H^1(E, G^{\text{ad}})$.

(c) Each inner twisting $\varphi : G \rightarrow G'$ gives rise to a twisting $\varphi_X : X \rightarrow X'$, where X' is an algebraic variety over E equipped with an action of G' , and φ_X is a $G_{E^{\text{sep}}} \cong G'_{E^{\text{sep}}}$ -equivariant isomorphism $X_{E^{\text{sep}}} \xrightarrow{\sim} X'_{E^{\text{sep}}}$. Explicitly, X' is a twist of X by the image of the cocycle $\{c_\sigma\}_\sigma \subset G^{\text{ad}}$ in $\text{Aut}(X)$. In particular, for each $\sigma \in \Gamma$ we have ${}^\sigma\varphi_X = \varphi \circ c_\sigma$.

By construction, for each $x \in X$ and $g \in G$, we have $\varphi_X(g(x)) = \varphi(g)(\varphi_X(x))$.

(d) An inner twisting φ is called *trivial*, if $\text{inv}(\varphi) = 1$. Explicitly, φ is trivial if and only if there exists $g \in G(E^{\text{sep}})$ such that $\varphi \circ \text{Int } g$ induces an isomorphism $G \xrightarrow{\sim} G'$ over E . In particular, the identity map $\text{Id}_G : G \rightarrow G$ is a trivial inner twisting.

Definition 1.1.2. (a) Two points $x, x' \in X(E)$ are called *conjugate*, if there exists $g \in G(E)$ such that $x' = g(x)$.

(b) Let $\varphi : G \rightarrow G'$ be an inner twisting, and $\varphi_X : X \rightarrow X'$ the corresponding twisting. Elements $x \in X(E)$ and $x' \in X'(E)$ are called *E^{sep} -conjugate*, if there exists $g \in G(E^{\text{sep}})$ such that $x' = \varphi_X(g(x))$.

(c) When G and G_x (and hence also G' and $G'_{x'}$) are connected reductive groups, E^{sep} -conjugate x and x' are also called *stably conjugate*.

Remark 1.1.3. All of our examples will satisfy assumption (c). In this case, our notion of stable conjugacy generalizes the standard one (see [Ko3]).

1.1.4. Cohomological invariants. Let $x \in X(E)$ and $x' \in X'(E)$ be E^{sep} -conjugate elements. Denote by $G_{x,x'}$ the set of $g \in G(E^{\text{sep}})$ such that $x' = \varphi_X(g(x))$.

(a) Assume that $\varphi = \text{Id}_G$. Then for each $g \in G_{x,x'}$, the map $\sigma \mapsto g^{-1}\sigma g$ defines a cocycle of Γ in G_x . Moreover, the corresponding cohomology class $\text{inv}(x, x') \in H^1(E, G_x)$ is independent of g . Furthermore, the correspondence $x' \mapsto \text{inv}(x, x')$ gives a bijection between the set of conjugacy classes of $x' \in X(E)$ stably conjugate to x and $\text{Ker}[H^1(E, G_x) \rightarrow H^1(E, G)]$ (compare [Ko2, 4.1]).

(b) Let φ be general. Then for each $g \in G_{x,x'}$, the map $\sigma \mapsto g^{-1}(\varphi^{-1}\sigma\varphi)^\sigma g$ defines a cocycle of Γ in $G_x/Z(G) = (G^{\text{ad}})_x \subset \text{Int } G$. Moreover, the corresponding cohomology class $\overline{\text{inv}}(x, x') \in H^1(E, (G^{\text{ad}})_x)$ is independent of g . Furthermore, the correspondence $x' \mapsto \overline{\text{inv}}(x, x')$ gives a surjection from the set of conjugacy classes

of $x' \in X'(E)$ stably conjugate to x to the preimage of $\text{inv}(G, G') \in H^1(E, G^{\text{ad}})$ in $H^1(E, (G^{\text{ad}})_x)$.

When $\varphi = \text{Id}_G$, then $\overline{\text{inv}}(x, x')$ is the image of $\text{inv}(x, x')$ under the natural projection $H^1(E, G_x) \rightarrow H^1(E, (G^{\text{ad}})_x)$.

(c) For each $g \in G_{x, x'}$, the map $h \mapsto \varphi(ghg^{-1})$ defines an inner twisting $G_x \rightarrow G'_{x'}$. Moreover, the corresponding invariant $\text{inv}(G_x, G'_{x'}) \in H^1(E, (G_x)^{\text{ad}})$ is just the image of $\overline{\text{inv}}(x, x')$. In particular, $G'_{x'}$ is canonically identified with G_x , if G_x is abelian.

(d) Assume that G_x is abelian, and let $y \in X(E)$ and $y' \in X'(E)$ be E^{sep} -conjugates of x and x' . Then the identification $G'_{y'} = G'_{x'} = G_x = G_y$ identifies $\overline{\text{inv}}(y, y')$ with the product of $\overline{\text{inv}}(x, x')$ and the images of $\text{inv}(y, x)$ and $\text{inv}(x', y')$. Moreover, if $\varphi = \text{Id}_G$, the same identification identifies $\text{inv}(y, y')$ with the product $\text{inv}(y, x) \text{inv}(x, x') \text{inv}(x', y')$.

1.1.5. Generalization (compare [LS, (3.4)]). Let $\varphi : G \rightarrow G'$ be an inner twisting, X_1, \dots, X_k a k -tuple of algebraic varieties over E equipped with an action of G^{ad} , and X'_1, \dots, X'_k the corresponding inner twistings. Let $x_i \in X_i(E)$ and $x'_i \in X'_i(E)$ be E^{sep} -conjugate for each $i = 1, \dots, k$.

Choose representatives $\tilde{c}_\sigma \in G(E^{\text{sep}})$ of $c_\sigma = \varphi^{-1}\sigma\varphi \in G^{\text{ad}}(E^{\text{sep}})$ for all $\sigma \in \Gamma$ and choose elements $g_i \in G(E^{\text{sep}})$ such that $x'_i = \varphi_{X_i}(g_i(x_i))$ for $i = 1, \dots, k$. Then the map $\sigma \mapsto [(g_i^{-1}\tilde{c}_\sigma^\sigma g_i)_i] \in G^k/Z(G)$ gives a cocycle of Γ in $(\prod_i G_{x_i})/Z(G)$, independent of the choice of \tilde{c}_σ 's, and the corresponding cohomology class

$$\overline{\text{inv}}((x_1, x'_1); \dots; (x_k, x'_k)) \in H^1(E, (\prod_i G_{x_i})/Z(G))$$

of $[(g_i^{-1}\tilde{c}_\sigma^\sigma g_i)_i]$ is independent of the g_i 's.

Note that $\overline{\text{inv}}((x_1, x'_1); \dots; (x_k, x'_k))$ lifts both $(\overline{\text{inv}}(x_i, x'_i)_i) \in \prod_i H^1(E, G_{x_i}/Z(G))$ and $\Delta(\text{inv}(G, G')) \in H^1(E, (G^k)/Z(G))$. (Here $\Delta : G^{\text{ad}} \rightarrow (G^k)/Z(G)$ is the diagonal embedding.)

The following result follows immediately from definitions.

Lemma 1.1.6. (a) $\overline{\text{inv}}((x_1, x'_1); \dots; (x_k, x'_k))$ depends only on the conjugacy classes of x_i 's and x'_i 's.

(b) If $\varphi = \text{Id}_G$, then $\overline{\text{inv}}((x_1, x'_1); \dots; (x_k, x'_k))$ is the image of $((\text{inv}(x_i, x'_i)_i) \in H^1(E, \prod_i G_{x_i})$.

(c) The canonical projection $(\prod_{i=1}^k G_{x_i})/Z(G) \rightarrow (\prod_{i=1}^{k-1} G_{x_i})/Z(G)$ maps $\overline{\text{inv}}((x_1, x'_1); \dots; (x_k, x'_k))$ to $\overline{\text{inv}}((x_1, x'_1); \dots; (x_{k-1}, x'_{k-1}))$.

(d) The diagonal map $G_x/Z(G) \hookrightarrow (G_x)^2/Z(G)$ maps $\overline{\text{inv}}(x, x') \in H^1(E, G_x/Z(G))$ to $\overline{\text{inv}}((x, x'); (x, x')) \in H^1(E, (G_x)^2/Z(G))$.

(e) Assume that each G_{x_i} is abelian, $y_i \in X_i(E)$ is an E^{sep} -conjugate of x_i , and $y'_i \in X'_i(E)$ is an E^{sep} -conjugate of x'_i . Then identifications $G_{x_i} = G_{y_i} = G_{x'_i}$ identify $\overline{\text{inv}}((y_1, y'_1); \dots; (y_k, y'_k))$ with the product of $\overline{\text{inv}}((x_1, x'_1); \dots; (x_k, x'_k))$ and the images of $((\text{inv}(y_i, x_i))_i) \in H^1(E, \prod_i G_{y_i})$ and $((\text{inv}(x'_i, y'_i))_i) \in H^1(E, \prod_i G_{x'_i})$.

(f) Let $\varphi' : G' \rightarrow G''$ be another inner twisting, $\varphi'_{X'} : X' \rightarrow X''$ the corresponding twisting of X' , and $x''_i \in X''(E)$ a stable conjugate of x_i and x'_i for each $i = 1, \dots, k$. If each G_{x_i} is abelian, then identifications $G_{x_i} = G_{x'_i}$ identify $\overline{\text{inv}}((x_1, x'_1); \dots; (x_k, x'_k)) \overline{\text{inv}}((x_1, x''_1); \dots; (x_k, x''_k))^{-1}$ with $\overline{\text{inv}}((x'_1, x''_1); \dots; (x'_k, x''_k))$.

Notation 1.1.7. (a) For each $x \in X(E)$, we denote by $[x] \subset X(E)$ the E^{sep} -conjugate class of x and by $a_x : G_x \hookrightarrow G$ the corresponding inclusion map.

(b) When G is reductive and connected, we denote by X^{sr} the set of $x \in X$ such that $G_x \subset G$ is a maximal torus and $\mathcal{G}_x \subset \mathcal{G}$ is a Lie algebra of G_x . We will call elements of X^{sr} *strongly regular*.

Remark 1.1.8. The condition on \mathcal{G}_x holds automatically if the characteristic of E is zero.

From now on we will assume that G is reductive and connected, and T is a torus over E of the same absolute rank as G .

1.1.9. Embedding of tori. (a) There exists an affine variety $\underline{\text{Emb}}(T, G)$ over E equipped with an action of G^{ad} such that for every extension E'/E the set $\underline{\text{Emb}}(T, G)(E')$ classifies embeddings $T_{E'} \hookrightarrow G_{E'}$, and G acts by conjugation.

To show the assertion, note that both G and T split over E^{sep} , therefore there exists an embedding $\iota : T_{E^{\text{sep}}} \hookrightarrow G_{E^{\text{sep}}}$. Consider the affine variety $\underline{\text{Emb}}_{\iota}(T, G) := G \times_{\text{Norm}_G(\iota(T))} \text{Aut}(T)$ over E^{sep} . Then the map $[g, \alpha] \mapsto (\text{Int } g) \circ \iota \circ \alpha$ defines a $G(E')$ -equivariant bijection $\psi_{\iota} : \underline{\text{Emb}}_{\iota}(T, G)(E') \xrightarrow{\sim} \underline{\text{Emb}}(T, G)(E')$ for every extension E' of E^{sep} (compare (the proof of) Lemma 1.1.10 (a) below).

For every two embeddings $\iota_1, \iota_2 : T_{E^{\text{sep}}} \hookrightarrow G_{E^{\text{sep}}}$, there exists a unique isomorphism $\underline{\text{Emb}}_{\iota_1}(T, G) \xrightarrow{\sim} \underline{\text{Emb}}_{\iota_2}(T, G)$, compatible with the ψ_{ι_j} 's, and we define $\underline{\text{Emb}}(T, G)$ be the inverse limit of the $\underline{\text{Emb}}_{\iota}(T, G)$'s. Finally, since $\underline{\text{Emb}}_{\iota}(T, G)$ is a disjoint union of affine varieties, it descends to E .

(b) For each $a \in \underline{\text{Emb}}(T, G)$, the stabilizer $G_a = a(T)$ is a maximal torus of G , which we will identify with T . It follows that $\underline{\text{Emb}}(T, G)^{\text{sr}} = \underline{\text{Emb}}(T, G)$. Also if $\varphi : G \rightarrow G'$ is an inner twisting, then the corresponding inner twisting $\underline{\text{Emb}}(T, G)'$ of $\underline{\text{Emb}}(T, G)$ is naturally isomorphic to $\underline{\text{Emb}}(T, G')$. In particular, we can speak about stable conjugacy of embeddings $a : T \hookrightarrow G$ and $a' : T \hookrightarrow G'$.

(c) If $x \in X^{\text{sr}}(E)$ and $x' \in X'^{\text{sr}}(E)$ are stably conjugate, then $a_x : G_x \hookrightarrow G$ and $a_{x'} : G_x \cong G'_{x'} \hookrightarrow G'$ are stably conjugate. Moreover, $\overline{\text{inv}}(a_x, a_{x'}) = \overline{\text{inv}}(x, x')$, (and $\text{inv}(a_x, a_{x'}) = \text{inv}(x, x')$ when $\varphi = \text{Id}_G$.)

Conversely, for every stably conjugate embeddings $a : T \hookrightarrow G$ and $a' : T \hookrightarrow G'$ and each $t \in T(E)$, elements $a(t) \in G(E)$ and $a'(t) \in G'(E)$ are E^{sep} -conjugate.

Lemma 1.1.10. (a) *Every conjugacy class $[a]$ of embeddings $T_{\overline{E}} \hookrightarrow G_{\overline{E}}$ contains an E^{sep} -rational embedding a .*

(b) *If G is quasi-split, then every Γ -invariant conjugacy class $[a]$ of embeddings $T_{\overline{E}} \hookrightarrow G_{\overline{E}}$ contains an E -rational embedding $a : T \hookrightarrow G$.*

(c) *Let $\varphi : G \rightarrow G'$ be an inner twisting such that G' is quasi-split. Then for every embedding $a : T \hookrightarrow G$ there exists an embedding $a' : T \hookrightarrow G'$ stably conjugate to a .*

Proof. (a) Let $S \subset G$ be a maximal torus over E . Then there exists $a \in [a]$ such that $a(T_{\overline{E}}) = S_{\overline{E}}$. Since both T and S split over E^{sep} , we get that a is E^{sep} -rational.

(b) When E is perfect, the assertion was shown in [Ko3, Cor. 2.2]. In general, the proof is similar: Since there is a Γ -equivariant bijection between maximal tori of G and those of G^{sc} , we can assume that G is semisimple and simply connected. Next fix an E^{sep} -rational $a' \in [a]$, which exists by (a). Since every homogeneous space for a connected group over a finite field has a rational point, the assertion holds in this case. Thus we may assume that E is infinite, therefore there exists $t \in T(E)$ such that $a'(t) \in G(E^{\text{sep}})$ is strongly regular.

The conjugacy class of $a'(t) \in G(E^{\text{sep}})$ is E^{sep} -rational and Γ -invariant. Thus it is E -rational. By the theorem of Steinberg [St, Thm 1.7] (when E is perfect) and Borel and Springer [BS, 8.6] (in the general case) there exists $g \in G(E^{\text{sep}})$ such that $ga'(t)g^{-1} \in G(E)$. Then $ga'(t)g^{-1} \in G^{\text{sr}}(E)$, hence $a := ga'g^{-1} : T \hookrightarrow G$ is E -rational.

(c) Since φ is an inner twisting, the conjugacy class of $\varphi \circ a : T_{\overline{E}} \hookrightarrow G'_{\overline{E}}$ is Γ -invariant. Hence by (b), $[\varphi \circ a]$ contains an E -rational element a' , which by definition is stably conjugate to a . \square

Corollary 1.1.11. *Let $\varphi : G \rightarrow G'$ be an inner twisting, and $\varphi_X : X \rightarrow X'$ the corresponding twisting. If G' is quasi-split, then for every $x \in X^{\text{sr}}(E)$ there exists $x' \in X'^{\text{sr}}(E)$ stably conjugate to x .*

Proof. Denote by $\iota : G/G_x \hookrightarrow X$ the canonical G -equivariant embedding $[g] \mapsto g(x)$, and by $\iota' : (G/G_x)' \hookrightarrow X'$ the twisted map. By Lemma 1.1.10 (c), there exists an embedding $a'_x : G_x \hookrightarrow G'$ stably conjugate to $a_x : G_x \hookrightarrow G$. Moreover, $(G/G_x)'$ is G' -equivariantly isomorphic to $G'/a'_x(G_x)$. It follows that the image of $[1] \in G'/a'_x(G_x)(E)$ under ι' is stably conjugate to x . \square

Definition 1.1.12. By a *quasi-isogeny* we call a homomorphism $\pi : \tilde{G} \rightarrow G$ such that $\pi(Z(\tilde{G})) \subset Z(G)$ and the induced homomorphism $\pi^{\text{ad}} : \tilde{G}^{\text{ad}} \rightarrow G^{\text{ad}}$ is an isomorphism.

1.1.13. Quasi-isogenies. Let $\pi : \tilde{G} \rightarrow G$ be a quasi-isogeny.

(a) Each inner twisting $\varphi : G \rightarrow G'$ gives rise to an inner twisting $\tilde{\varphi} : \tilde{G} \rightarrow \tilde{G}'$ such that $\text{inv}(\tilde{G}, \tilde{G}') = \text{inv}(G, G')$ (in $H^1(E, \tilde{G}^{\text{ad}}) = H^1(E, G^{\text{ad}})$).

(b) X is equipped with an action of \tilde{G} trivial on $Z(\tilde{G})$.

(c) There is a π - and Γ -equivariant bijection between embeddings of maximal tori $a : T \hookrightarrow G$ and the corresponding embeddings $\tilde{a} : \tilde{T} \hookrightarrow \tilde{G}$. Indeed, given a , put $\tilde{T} = T \times_G \tilde{G} := \{t \in T, \tilde{g} \in \tilde{G} \mid a(t) = \pi(\tilde{g})\}$ and $\tilde{a}(t, \tilde{g}) := \tilde{g}$. Conversely, given \tilde{a} , define a be the embedding $\pi(\tilde{a}(\tilde{T})) \hookrightarrow G$. In particular, a_1 and a_2 are stably conjugate if and only if \tilde{a}_1 and \tilde{a}_2 are such.

We will call \tilde{a} the *lift* of a and $[\tilde{a}]$ the *lift* of $[a]$.

(d) For each $i = 1, \dots, k$, let $a_i : T_i \hookrightarrow G$ and $a'_i : T_i \hookrightarrow G'$ be stable conjugate embeddings of maximal tori, and let $\tilde{a}_i : \tilde{T}_i \hookrightarrow \tilde{G}$ and $\tilde{a}'_i : \tilde{T}_i \hookrightarrow \tilde{G}'$ be the lifts of the a_i 's and the a'_i 's, respectively. Then $\overline{\text{inv}}((a_1, a'_1); \dots; (a_k, a'_k)) \in H^1(E, (\prod_i T_i)/Z(G))$ is the image of $\overline{\text{inv}}((\tilde{a}_1, \tilde{a}'_1); \dots; (\tilde{a}_k, \tilde{a}'_k)) \in H^1(E, (\prod_i \tilde{T}_i)/Z(\tilde{G}))$ under the canonical map $(\prod_i \tilde{T}_i)/Z(\tilde{G}) \rightarrow (\prod_i T_i)/Z(G)$.

1.2. Preliminaries on dual groups.

In this subsection, we will recall basic properties of Langlands dual groups. More specifically, we will study properties of triples $(G, H, [\eta])$ from 1.2.3. Constructions from this subsection will be later used in the case when H is an endoscopic group for G .

Notation 1.2.1. For each connected reductive group G over a field E , we denote by \widehat{G} (or $\{G\}^\vee$) the complex connected Langlands dual group, and by $\rho_G : \Gamma \rightarrow \text{Out}(\widehat{G})$ the corresponding Galois action.

1.2.2. Basic properties of dual groups. (a) The map $G \mapsto (\widehat{G}, \rho_G)$ defines a surjection from the set of isomorphism classes of connected reductive groups over E to that of pairs consisting of a connected complex reductive group \widehat{G} , and a continuous homomorphism $\rho : \Gamma \rightarrow \text{Out}(\widehat{G})$. Moreover, $(\widehat{G}_1, \rho_{G_1}) \cong (\widehat{G}_2, \rho_{G_2})$ if and only if G_2 is an inner twist of G_1 . In particular, each pair (\widehat{G}, ρ_G) comes from a unique quasi-split group G over E .

(b) Let T be a torus over E of the same absolute rank as G . Then there exists a canonical (hence Γ -equivariant) bijection $[a] \mapsto [\tilde{a}]$ between conjugacy classes of embeddings $T_{\overline{E}} \hookrightarrow G_{\overline{E}}$ and conjugacy classes of embeddings $\widehat{T} \hookrightarrow \widehat{G}$. In particular, $[a]$ is Γ -invariant if and only if $[\tilde{a}]$ is such.

(c) For each embeddings of maximal tori $T_{\overline{E}} \hookrightarrow G_{\overline{E}}$ and $\widehat{T} \hookrightarrow \widehat{G}$ related as in (b), the set of roots (resp. coroots) of $(G_{\overline{E}}, T_{\overline{E}})$ is canonically identified with the set of

coroots (resp. roots) of $(\widehat{G}, \widehat{T})$. In particular, the Weyl group $W(\widehat{G})$ is canonically identified with $W(G)$.

(d) Each quasi-isogeny $\pi : G_1 \rightarrow G_2$ gives rise to a conjugacy class $[\widehat{\pi}]$ of quasi-isogenies $\widehat{G}_2 \rightarrow \widehat{G}_1$. In particular, it induces a homomorphism $Z_{[\widehat{\pi}]} : Z(\widehat{G}_2) \rightarrow Z(\widehat{G}_1)$.

1.2.3. Triple. For the rest of this subsection, we fix a triple $(G, H, [\eta])$, consisting of a connected reductive group G over a field E , a quasi-split reductive group H over E of the same absolute rank as G , and a Γ -invariant \widehat{G} -conjugacy class $[\eta]$ of embeddings $\widehat{H} \hookrightarrow \widehat{G}$.

1.2.4. Properties of the triple $(G, H, [\eta])$. (a) Every stable conjugacy class $[b]$ of embeddings of maximal tori $T \hookrightarrow H$ defines a Γ -invariant conjugacy class $[\widehat{b}]$ of embeddings $\widehat{T} \hookrightarrow \widehat{H}$ (by 1.2.2), hence a Γ -invariant conjugacy class $[\widehat{b}]_G := [\eta] \circ [\widehat{b}]$ of embeddings $\widehat{T} \hookrightarrow \widehat{G}$, thus a Γ -invariant conjugacy class $[b]_G$ of embeddings $T_{\overline{E}} \hookrightarrow G_{\overline{E}}$.

(b) There exist canonical (Γ -equivariant) embeddings $Z(\widehat{G}) \hookrightarrow Z(\widehat{H})$, $Z(G) \hookrightarrow Z(H)$ and $W(H) \hookrightarrow W(G)$.

To see it, fix a maximal torus $\widehat{T} \subset \widehat{H}$ and an embedding $\eta : \widehat{H} \hookrightarrow \widehat{G}$ from $[\eta]$. Then \widehat{T} is a maximal torus of \widehat{G} , hence the set of roots (therefore also of coroots) of $(\widehat{H}, \widehat{T})$ is naturally a subset of that of $(\widehat{G}, \widehat{T})$. It follows that $W(H) = W(\widehat{H})$ is naturally a subgroup of $W(\widehat{G}) = W(G)$. Also by 1.2.2 (c), the set of roots of (H, T) is naturally a subset of that of (G, T) . Since $Z(\widehat{G}) \subset \widehat{T}$ (resp. $Z(G) \subset T$) is the intersection of kernels of all roots of $(\widehat{G}, \widehat{T})$ (resp. of (G, T)), and similarly for $Z(\widehat{H})$ (resp. $Z(H)$), we get an embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H})$ (resp. $Z(G) \hookrightarrow Z(H)$).

(c) $[\eta]$ naturally gives rise to a conjugacy class $[\overline{\eta}]$ of embeddings $\widehat{H}/Z(\widehat{G}) \hookrightarrow \widehat{G}^{\text{ad}}$. Namely, each $\eta : \widehat{H} \hookrightarrow \widehat{G}$ from $[\eta]$, has a unique lift $\overline{\eta} : \widehat{H}/Z(\widehat{G}) \hookrightarrow \widehat{G}^{\text{ad}}$, and we denote by $[\overline{\eta}]$ the corresponding conjugacy class.

By (b), $[\eta]$ thus induces a homomorphism $Z_{[\overline{\eta}]} : Z(\widehat{G}^{\text{ad}})^{\Gamma} \rightarrow \pi_0(Z(\widehat{H}/Z(\widehat{G}))^{\Gamma})$.

1.2.5. Example. Any embedding $a : T \hookrightarrow G$ of a maximal torus gives rise to a triple 1.2.3 with $H = T$ and $[\eta] = [\widehat{a}]$. In particular, it gives rise to a Γ -equivariant embedding $Z(\widehat{G}) \hookrightarrow \widehat{T}$, hence to a homomorphism $\pi_0(Z(\widehat{G})^{\Gamma}) \rightarrow \pi_0(\widehat{T}^{\Gamma})$. We will denote both of these maps by $Z_{[\widehat{a}]}$.

1.2.6. Construction of an exact sequence. For each maximal torus $T \subset H$, we consider the exact sequence

$$(1.2.1) \quad 0 \rightarrow \widehat{T} \xrightarrow{\mu_T} \widehat{T^2/Z(G)} \xrightarrow{\nu_T} \widehat{T/Z(G)} \rightarrow 0,$$

dual to the exact sequence $0 \rightarrow T/Z(G) \xrightarrow{\widehat{\nu}_T} T^2/Z(G) \xrightarrow{\widehat{\mu}_T} T \rightarrow 0$, where $\widehat{\nu}_T$ is the diagonal morphism and $\widehat{\mu}_T([t_1, t_2]) := t_1/t_2$.

Since a center of a reductive group equals to the intersection of kernels of all roots, we conclude that $\mu_T(Z(\widehat{H})) \subset Z(\widehat{H^2/Z(G)})$, $\nu_T(Z(\widehat{H^2/Z(G)})) \subset Z(\widehat{H/Z(G)})$, and the induced sequence

$$(1.2.2) \quad 0 \rightarrow Z(\widehat{H}) \xrightarrow{\mu_H} Z(\widehat{H^2/Z(G)}) \xrightarrow{\nu_H} Z(\widehat{H/Z(G)})$$

is Γ -equivariant and exact. Furthermore, since over \overline{E} all maximal tori of H are conjugate, sequence (1.2.2) is independent of the choice of T .

Observe that the composition of μ_H with the projection $Z(\widehat{H^2/Z(G)}) \rightarrow Z(\widehat{H^2}) = Z(\widehat{H})^2$ is the map $z \mapsto (z, z^{-1})$.

Lemma 1.2.7. *For each $i, j \in \{1, 2, 3\}$, let $\mu_{i,j} : Z(\widehat{H}) \hookrightarrow Z(\widehat{H^3/Z(G)})$ be the composition of μ_H and the embedding $Z(\widehat{H^2/Z(G)}) \hookrightarrow Z(\widehat{H^3/Z(G)})$ corresponding to the projection $H^3/Z(G) \rightarrow H^2/Z(G)$ to the i -th and the j -th factors. Then $\mu_{1,3} = \mu_{1,2}\mu_{2,3}$.*

Proof. Consider the projection $\lambda_{i,j} : T^3/Z(G) \rightarrow T$ given by the rule $\lambda_{i,j}([t_1, t_2, t_3]) = t_j/t_i$. Since each $\mu_{i,j}$ is the restriction of $\lambda_{i,j} : \widehat{T} \hookrightarrow \widehat{T^3/Z(G)}$ to $Z(\widehat{H})$, the equality $\mu_{1,3} = \mu_{1,2}\mu_{2,3}$ follows from the equality $\lambda_{1,3} = \lambda_{1,2}\lambda_{2,3}$. \square

1.2.8. Construction of two homomorphisms. (a) Consider a pair $[b_i]$ of stable conjugacy classes of embeddings of maximal tori $T_i \hookrightarrow H$. Then the $[b_i]$'s give rise to a stable conjugacy class $[b_1, b_2]$ of embeddings $(T_1 \times T_2)/Z(G) \hookrightarrow H^2/Z(G)$, hence to a Γ -invariant embedding

$$(1.2.3) \quad \iota([b_1], [b_2]) := Z(\widehat{[b_1, b_2]}) \circ \mu_H : Z(\widehat{H}) \hookrightarrow \{(T_1 \times T_2)/Z(G)\}^\sim$$

In its turn, $\iota([b_1], [b_2])$ induces a homomorphism

$$(1.2.4) \quad \kappa([b_1], [b_2]) : \pi_0(Z(\widehat{H})^\Gamma) \rightarrow \pi_0(\{(T_1 \times T_2)/Z(G)\}^\sim^\Gamma),$$

(b) Assume that in the notation of (a), we have $T_1 = T_2 = T$ and $[b_1]_G = [b_2]_G$. Then $\iota([b_1]_G, [b_2]_G) = \iota([b_1], [b_2])|_{Z(\widehat{G})}$ factors through $\mu_T : \widehat{T} \hookrightarrow \widehat{T^2/Z(G)}$, hence the image of $\iota([b_1]_G, [b_2]_G)$ lies in $\text{Ker } \nu_T$. Thus the composition $\nu_T \circ \iota([b_1], [b_2]) : Z(\widehat{H}) \rightarrow \widehat{T/Z(G)}$ factors through $Z(\widehat{H})/Z(\widehat{G})$ and induces a homomorphism

$$(1.2.5) \quad \kappa \left(\begin{bmatrix} [b_1] \\ [b_2] \end{bmatrix} \right) : \pi_0(Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma) \rightarrow \pi_0(\widehat{T/Z(G)})^\Gamma.$$

(c) For each $i = 1, 2$, denote by $[\bar{b}_i]$ the stable conjugacy class of embeddings $T/Z(G) \hookrightarrow H/Z(G)$ induced by $[b_i]$. Then the composition of the projection $Z(\widehat{H/Z(G)}) \rightarrow Z(\widehat{H})$ and $\nu_T \circ \iota([b_1], [b_2]) : Z(\widehat{H}) \rightarrow T/Z(G)$ equals the quotient $Z_{[\bar{b}_1]}/Z_{[\bar{b}_2]} : Z(\widehat{H/Z(G)}) \rightarrow T/Z(G)$.

This gives the following description of the map $\kappa \left(\frac{[b_1]}{[b_2]} \right)$. For each representative $\tilde{s} \in Z(\widehat{H/Z(G)})$ of $\bar{s} \in \pi_0(Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma)$, the quotient $Z_{[\bar{b}_1]}(\tilde{s})/Z_{[\bar{b}_2]}(\tilde{s}) \in T/Z(G)$ is Γ -invariant, and $\kappa \left(\frac{[b_1]}{[b_2]} \right) (\bar{s})$ equals the class of $Z_{[\bar{b}_1]}(\tilde{s})/Z_{[\bar{b}_2]}(\tilde{s})$. (Note that $\tilde{s} \in Z(\widehat{H/Z(G)})$ is not always Γ -invariant).

Lemma 1.2.9. (a) Let $[b_1]$ and $[b_2]$ be as in 1.2.8, and let $\check{s} \in \pi_0(Z(\widehat{H})^\Gamma)$ be a representative of $\bar{s} \in \pi_0(Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma)$. Then the quotient $Z_{[\bar{b}_1]}(\check{s})/Z_{[\bar{b}_2]}(\check{s}) \in \pi_0(\widehat{T}^\Gamma)$ equals the image of $\kappa \left(\frac{[b_1]}{[b_2]} \right) (\bar{s})$.

(b) Let $[b_1], [b_2]$ and $[b_3]$ be stable conjugacy classes of embeddings of maximal tori $T \hookrightarrow H$ such that $[b_1]_G = [b_2]_G = [b_3]_G$. Then we have $\kappa \left(\frac{[b_1]}{[b_3]} \right) = \kappa \left(\frac{[b_1]}{[b_2]} \right) \kappa \left(\frac{[b_2]}{[b_3]} \right)$.

Proof. Both assertions follow from the description of $\kappa \left(\frac{[b_i]}{[b_j]} \right)$, given in 1.2.8 (c). \square

1.3. Endoscopic triples: basic properties.

Let G be a connected reductive group over a field E . In this subsection we give basic constructions and properties of endoscopic triples (compare [Ko1, §7]).

Definition 1.3.1. (a) A triple $\mathcal{E} = (H, [\eta], \bar{s})$ consisting of

- a quasi-split reductive group H over E of the same absolute rank as G ;
- a Γ -invariant \widehat{G} -conjugacy class $[\eta]$ of embeddings $\widehat{H} \hookrightarrow \widehat{G}$;
- an element $\bar{s} \in \pi_0(Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma)$, where the Γ -equivariant embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H})$ is induced by $[\eta]$ (see 1.2.4 (b)),

is called an *endoscopic triple* for G , if for a generic representative $s \in Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma$ of \bar{s} we have $\eta(\widehat{H}) = \widehat{G}_{\eta(s)}$ for all $\eta \in [\eta]$. Such a representative s we will call \mathcal{E} -*compatible*.

(b) An endoscopic triple \mathcal{E} for G is called *elliptic*, if the group $Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma$ is finite.

(c) An *isomorphism* from an endoscopic triple $\mathcal{E}_1 = (H_1, [\eta_1], \bar{s}_1)$ to $\mathcal{E}_2 = (H_2, [\eta_2], \bar{s}_2)$ is an isomorphism $\alpha : H_1 \xrightarrow{\sim} H_2$ such that the corresponding element $[\alpha]$ of $\text{Isom}(\widehat{H}_2, \widehat{H}_1)/\text{Int}(\widehat{H}_2)$ satisfies $[\eta_1] \circ [\alpha] = [\eta_2]$ and $\bar{s}_1 = [\alpha](\bar{s}_2)$.

(d) We denote by $\text{Aut}(\mathcal{E})$ the group of automorphisms of \mathcal{E} and by $\Lambda(\mathcal{E})$ the quotient $\text{Aut}(\mathcal{E})/H^{\text{ad}}(E)$.

(e) An endoscopic triple \mathcal{E} is called *split* (resp. *unramified*), if H is a split (resp. unramified) group over E .

Remark 1.3.2. (a) When E is a local non-archimedean field, our notion of an endoscopic triple is equivalent to the standard one ([Ko1, 7.4]). Indeed, let (H, η, s) be an endoscopic triple in the sense of [Ko1, 7.4]. Since $Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma$ is a subgroup of finite index in $[Z(\widehat{H})/Z(\widehat{G})]^\Gamma$, condition [Ko1, 7.4.3] asserts that the image of $s \in Z(\widehat{H})$ in $Z(\widehat{H})/Z(\widehat{G})$ belongs to $Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma$, thus defines an element $\bar{s} \in \pi_0(Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma)$. Moreover, the map $(H, \eta, s) \mapsto (H, [\eta], \bar{s})$ defines an equivalence of categories between endoscopic triples in the sense of [Ko1, 7.4] and those in our sense.

If (H, η_1, s_1) and (H, η_2, s_2) are two endoscopic triples in the sense of [Ko1, 7.4] such that $[\eta_1] = [\eta_2]$ and $\bar{s}_1 = \bar{s}_2$, then they are canonically isomorphic. Therefore we have chosen to deviate from the Kottwitz' notation and to identify these objects.

(b) When E is a number field, then our notion of an endoscopic triple slightly differs from the standard one ([Ko1, 7.4]). However in this paper we will only consider the case $G = G^{\text{sc}}$ (see Appendix B). In this case, $Z(\widehat{G}) = 1$, hence both notions coincide.

Remark 1.3.3. Since an inner twisting $\varphi : G \rightarrow G'$ identifies the dual groups of G and G' , every endoscopic triple for G defines the one for G' .

Remark 1.3.4. In the notation of Definition 1.3.1 (c), the map $[\alpha] \mapsto [\widehat{\alpha}]$ identifies $\text{Isom}(\mathcal{E}_1, \mathcal{E}_2)/H_1^{\text{ad}}(E)$ with the set of all Γ -invariant $[\widehat{\alpha}] \in \text{Isom}(\widehat{H}_2, \widehat{H}_1)/\text{Int}(\widehat{H}_2)$ such that $[\eta_1] \circ [\widehat{\alpha}] = [\eta_2]$ and $\bar{s}_1 = [\widehat{\alpha}](\bar{s}_2)$.

In particular, for each endoscopic triple $\mathcal{E} = (H, [\eta], \bar{s})$, the map $[\alpha] \mapsto [\widehat{\alpha}]$ identifies $\Lambda(\mathcal{E})$ with the set of $g \in \text{Out}(\widehat{H})^\Gamma$ such that $g(\bar{s}) = \bar{s}$ and $[\eta] \circ g = [\eta]$.

1.3.5. Homomorphisms, corresponding to endoscopic triples. To every endoscopic triple $\mathcal{E} = (H, [\eta], \bar{s})$ for G , we are going to associate a homomorphism $\pi_{\mathcal{E}} : \Lambda(\mathcal{E}) \rightarrow \pi_0(Z(\widehat{H}/Z(\widehat{G}))^\Gamma)$. Moreover, the image of $\pi_{\mathcal{E}}$ lies in the image of the canonical map $Z_{[\eta]} : Z(\widehat{G}^{\text{ad}})^\Gamma \rightarrow \pi_0(Z(\widehat{H}/Z(\widehat{G}))^\Gamma)$ from 1.2.4 (c).

To define $\pi_{\mathcal{E}}$, fix $\eta \in [\eta]$, and identify \widehat{H} with its image $\eta(\widehat{H}) \subset \widehat{G}$. Then η lifts to an embedding $\widehat{H}/Z(\widehat{G}) \hookrightarrow \widehat{G}^{\text{ad}} = \widehat{G}^{\text{sc}}$. For each $\alpha \in \Lambda(\mathcal{E})$, choose $g \in \text{Norm}_{\widehat{G}^{\text{ad}}}(\widehat{H})$ inducing $\widehat{\alpha} \in \text{Out}(\widehat{H})$. Then g normalizes $\widehat{H}/Z(\widehat{G}) \subset \widehat{G}^{\text{sc}}$, hence it induces an element $\widetilde{\alpha} \in \text{Out}(\widehat{H}/Z(\widehat{G}))$. Moreover, $\widetilde{\alpha}$ is Γ -invariant and independent of the choices of η and g .

Furthermore, $\tilde{\alpha}$ is trivial on the kernel $\text{Ker}[\widehat{H/Z(G)} \rightarrow \widehat{H}/Z(\widehat{G})]$. Therefore for each $\tilde{t} \in Z(\widehat{H/Z(G)})$ the quotient $\tilde{\alpha}(\tilde{t})/\tilde{t} \in Z(\widehat{H/Z(G)})$ depends only on the image $t \in Z(\widehat{H})/Z(\widehat{G})$ of \tilde{t} . Hence the map $t \mapsto \tilde{\alpha}(\tilde{t})/\tilde{t}$ defines a Γ -equivariant homomorphism $Z(\widehat{H})/Z(\widehat{G}) \rightarrow Z(\widehat{H/Z(G)})$, which in its turn induces a homomorphism $\tilde{\alpha} : \pi_0(Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma) \rightarrow \pi_0(Z(\widehat{H/Z(G)})^\Gamma)$.

We define $\pi_{\mathcal{E}}$ by the formula $\pi_{\mathcal{E}}(\alpha) := \tilde{\alpha}(\bar{s}) \in \pi_0(Z(\widehat{H/Z(G)})^\Gamma)$. Since $\widehat{\alpha}(\bar{s}) = \bar{s}$, we see that $\pi_{\mathcal{E}}$ is a homomorphism, and the image of $\pi_{\mathcal{E}}$ lies in the kernel $\text{Ker}[\pi_0(Z(\widehat{H/Z(G)})^\Gamma) \rightarrow \pi_0(Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma)]$. As the last group coincides with the image of $Z_{[\eta]} : Z(\widehat{G^{\text{ad}}})^\Gamma \rightarrow \pi_0(Z(\widehat{H/Z(G)})^\Gamma)$ (see [Ko1, Cor. 2.3]), we get the assertion.

Lemma 1.3.6. *For every embedding of maximal torus $b : T \hookrightarrow H$ the corresponding map $Z_{[\hat{b}]} : \pi_0(Z(\widehat{H/Z(G)})^\Gamma) \rightarrow \pi_0(\widehat{T/Z(G)})^\Gamma$ maps $\pi_{\mathcal{E}}(\alpha)$ to $\kappa\left(\frac{\alpha([b])}{[b]}\right)(\bar{s})$.*

Proof. The assertion follows immediately from 1.2.8 (c). \square

Notation 1.3.7. Let $\mathcal{E} = (H, [\eta], \bar{s})$ be an endoscopic triple for G .

(a) Denote by $\Pi_{\mathcal{E}}$ the map sending a stable conjugacy class $[b]$ of embeddings of maximal tori $T \hookrightarrow H$ to a pair consisting of an E -rational conjugacy class $[b]_G$ (see 1.2.4 (c)) of embeddings $T_E \hookrightarrow G_E$ and an element $\bar{\kappa}_{[b]} := Z_{[\hat{b}]}(\bar{s}) \in \pi_0(\widehat{T^\Gamma/Z(\widehat{G})^\Gamma})$.

(b) Denote by $Z(\mathcal{E}) \subset Z(\widehat{G^{\text{ad}}})^\Gamma$ the preimage of $\text{Im } \pi_{\mathcal{E}} \subset \pi_0(Z(\widehat{H/Z(G)})^\Gamma)$ under $Z_{[\eta]}$.

(c) For each pair $([a], \bar{\kappa}) \in \text{Im } \Pi_{\mathcal{E}}$, we denote by $S_{([a], \bar{\kappa})} \subset \pi_0(\widehat{T/Z(G)})^\Gamma$ the subgroup consisting of elements $\kappa\left(\frac{[b_1]}{[b_2]}\right)(\bar{s})$, where b_1 and b_2 run through the set of embeddings $b : T \hookrightarrow H$ such that $\Pi_{\mathcal{E}}([b]) = ([a], \bar{\kappa})$. We denote by $Z(\mathcal{E}, [a], \bar{\kappa}) \subset Z(\widehat{G^{\text{ad}}})^\Gamma$ the preimage of $S_{([a], \bar{\kappa})}$ under the map $Z_{[\hat{a}]} : Z(\widehat{G^{\text{ad}}})^\Gamma \rightarrow \pi_0(\widehat{T/Z(G)})^\Gamma$.

(d) We call embeddings of maximal tori $a : T \hookrightarrow G$ and $b : T \hookrightarrow H$ *compatible*, if $a \in [b]_G$.

1.3.8. Endoscopic triples, corresponding to pairs. (a) Following Langlands ([La1, II, 4]), we associate an endoscopic triple $\mathcal{E} = \mathcal{E}_{([a], \kappa)}$ for G to each pair $([a], \kappa)$, consisting of a stable conjugacy class of an embedding $a : T \hookrightarrow G$ of maximal torus and an element $\kappa \in \widehat{T^\Gamma/Z(\widehat{G})^\Gamma}$. Moreover, $\mathcal{E}_{([a], \kappa)}$ is elliptic, if $a(T)$ is an elliptic torus of G .

For the convenience of the reader, we will recall this important construction. Choose an element $\widehat{a} : \widehat{T} \hookrightarrow \widehat{G}$ of $[\widehat{a}]$, and identify \widehat{T} with $\widehat{a}(\widehat{T})$. Choose a representative $\tilde{\kappa} \in \widehat{T}^\Gamma$ of κ , and put $s := \widehat{a}(\tilde{\kappa}) \in \widehat{G}/Z(\widehat{G})$, $\widehat{H} := \widehat{G}_s^0$, and let $[\eta]$ be the conjugacy class of the inclusion $\widehat{H} \hookrightarrow \widehat{G}$.

Then the image of $\rho_T : \Gamma \rightarrow \text{Out}(\widehat{T})$ lies in $\text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{T})/\widehat{T}$, and $\rho_G : \Gamma \rightarrow \text{Out}(\widehat{G})$ is the composition of $\rho_T : \Gamma \rightarrow \text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{T})/\widehat{T}$ with the canonical homomorphism $\text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{T})/\widehat{T} \hookrightarrow \text{Aut}(\widehat{G}) \rightarrow \text{Out}(\widehat{G})$.

As $\tilde{\kappa}$ belongs to \widehat{T}^Γ , the image of ρ_T lies in $\text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{T})_s/\widehat{T}$, where $\text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{T})_s$ is the stabilizer of s in $\text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{T})$. Since $\text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{T})_s \subset \text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{H})$, we can compose $\rho_T : \Gamma \rightarrow \text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{T})_s/\widehat{T}$ with the canonical homomorphism $\text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{T})_s/\widehat{T} \rightarrow \text{Norm}_{\text{Aut}(\widehat{G})}(\widehat{H})/\widehat{H} \subset \text{Out}(\widehat{H})$. We denote the composition $\Gamma \rightarrow \text{Out}(\widehat{H})$ by ρ , and let H be the unique quasi-split group over E corresponding to the pair (\widehat{H}, ρ) (see 1.2.2 (a)).

By construction, $[\eta]$ is Γ -equivariant, and s belongs to $Z(\widehat{H})^\Gamma$. Denote by $\bar{s} \in \pi_0(Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma)$ the class of s . Then $(H, [\eta], \bar{s})$ is an endoscopic triple for G , independent of the choices of \widehat{a} and $\tilde{\kappa}$. We denote this endoscopic triple by $\mathcal{E}_{([a], \kappa)}$.

(b) In the notation of (a), we have $([a], \bar{\kappa}) \in \text{Im } \Pi_{\mathcal{E}}$.

To find $[b] \in \Pi_{\mathcal{E}}^{-1}([a], \bar{\kappa})$, we choose \widehat{a} and $\tilde{\kappa}$ as in (a) and identify \widehat{T} with $\widehat{a}(\widehat{T})$. By construction, there exists $\eta \in [\eta]$ such that $\widehat{T} \subset \eta(\widehat{H})$ and $\eta(s) = \tilde{\kappa}$. Let $[\widehat{b}]$ be the conjugacy class of the inclusion $\widehat{T} \hookrightarrow \eta(\widehat{H}) \cong \widehat{H}$. Then $[\widehat{b}]$ is Γ -equivariant, hence it gives rise to a stable conjugacy class $[b]$ of embeddings $T \hookrightarrow H$ (see Lemma 1.1.10 (b)), which belongs to $\Pi_{\mathcal{E}}^{-1}([a], \bar{\kappa})$.

Lemma 1.3.9. *Let $\mathcal{E} = (H, [\eta], \bar{s})$ be an endoscopic triple for G , $b : T \hookrightarrow H$ and $a : T \hookrightarrow G$ embeddings of maximal tori, $\bar{\kappa}$ an element of $\pi_0(\widehat{T}^\Gamma/Z(\widehat{G})^\Gamma)$ such that $\Pi_{\mathcal{E}}([b]) = ([a], \bar{\kappa})$.*

(a) *There exists a representative $\kappa \in \widehat{T}^\Gamma/Z(\widehat{G})^\Gamma$ of $\bar{\kappa}$ such that $\mathcal{E} \cong \mathcal{E}_{([a], \kappa)}$.*

(b) *The group $Z(\mathcal{E})$ is contained in $Z(\mathcal{E}, [a], \bar{\kappa})$. Moreover, if $b(T) \subset H$ is elliptic, then we have $Z(\mathcal{E}, [a], \bar{\kappa}) = Z(\mathcal{E})$. In particular, $Z(\mathcal{E}, [a], \bar{\kappa}) = Z(\mathcal{E})$ if $a(T) \subset G$ is elliptic.*

Proof. (a) Let $s \in Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma$ be an \mathcal{E} -compatible representative of \bar{s} , and let $\kappa \in \widehat{T}^\Gamma/Z(\widehat{G})^\Gamma$ be the image of s under the embedding $Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma \hookrightarrow \widehat{T}^\Gamma/Z(\widehat{G})^\Gamma$, induced by $[\widehat{b}]$. Then κ is a representative of $\bar{\kappa}$, and we claim that $\mathcal{E}_{([a], \kappa)} \cong \mathcal{E}$. To show it, choose an embedding $\widehat{b} : \widehat{T} \hookrightarrow \widehat{H}$ such that $\widehat{b} \in [\widehat{b}]$. Then for each $\eta \in [\eta]$, the composition $\widehat{a} := \eta \circ \widehat{b} : \widehat{T} \hookrightarrow \widehat{G}$ belongs to $[\widehat{a}]$ and satisfies $\eta(\widehat{H}) = \widehat{G}_{\widehat{a}(\kappa)}^0$.

Finally, since the conjugacy class of the embedding $\widehat{b} : \widehat{T} \hookrightarrow \widehat{H}$ is Γ -invariant, the homomorphism $\rho_H : \Gamma \rightarrow \text{Out}(\widehat{H})$ is induced by $\rho_T : \Gamma \rightarrow \text{Out}(\widehat{T})$ as in 1.3.8 (a).

(b) Since for each $\alpha \in \Lambda(\mathcal{E})$, the stable conjugacy class $\alpha([b])$ belongs to $\Pi_{\mathcal{E}}^{-1}([a], \overline{\kappa})$, the inclusion $Z(\mathcal{E}) \subset Z(\mathcal{E}, [a], \overline{\kappa})$ follows from Lemma 1.3.6.

Assume now that $b(T) \subset H$ is elliptic. Then for the second assertion, it will suffice to check that for each $[b'] \in \Pi_{\mathcal{E}}^{-1}([a], \overline{\kappa})$, there exists $\alpha \in \Lambda(\mathcal{E})$ such that $[b'] = \alpha([b])$.

Embed \widehat{H} into \widehat{G} by means of an element of $[\eta]$, choose embeddings $\widehat{b} : \widehat{T} \hookrightarrow \widehat{H} \subset \widehat{G}$ and $\widehat{b}' : \widehat{T} \hookrightarrow \widehat{H} \subset \widehat{G}$ from $[\widehat{b}]$ and $[\widehat{b}']$, respectively, and identify \widehat{T} with its image $\widehat{b}(\widehat{T}) \subset \widehat{H}$. Replacing \widehat{b}' by its \widehat{H} -conjugate, we may assume that $\widehat{b}'(\widehat{T}) = \widehat{b}(\widehat{T})$.

Choose a representative $s \in Z(\widehat{H})^\Gamma$ of \overline{s} such that $\widehat{H} = \widehat{G}_s^0$. Since $\Pi_{\mathcal{E}}([b']) = \Pi_{\mathcal{E}}([b])$, there exists an element $g \in \widehat{G}$ such that $\widehat{b}' = g\widehat{b}g^{-1}$ and $(g^{-1}sg)s^{-1}$ belongs to $(\widehat{T}^\Gamma)^0 Z(\widehat{G})^\Gamma = (Z(\widehat{H})^\Gamma)^0 Z(\widehat{G})^\Gamma \subset Z(\widehat{H})$. Therefore $g^{-1}sg \in Z(\widehat{H})$, hence $g\widehat{H}g^{-1} = \widehat{H}$. Let $\overline{g} \in \text{Out}(\widehat{H})$ be the class of g . Since the conjugacy classes of $\widehat{b} : \widehat{T} \hookrightarrow \widehat{H}$ and $\widehat{b}' : \widehat{T} \hookrightarrow \widehat{H}$ are Γ -invariant, we get from the equality $\widehat{b}' = g\widehat{b}g^{-1}$ that \overline{g} is Γ -invariant. In other words, there exists $\alpha \in \Lambda(\mathcal{E})$ such that $\widehat{\alpha} \in \text{Out}(\widehat{H})$ equals \overline{g} . Then $[b'] = \alpha([b])$, as claimed. \square

Lemma 1.3.10. *Let $\mathcal{E} = (H, [\eta], \overline{s})$ be an endoscopic triple for G , and let $\pi : \widetilde{G} \rightarrow G$ be a quasi-isogeny.*

(a) *There exists a unique pair consisting of an endoscopic triple $\widetilde{\mathcal{E}} = (\widetilde{H}, [\widetilde{\eta}], \widetilde{\overline{s}})$ for \widetilde{G} and a stable conjugacy class of quasi-isogenies $\pi' : \widetilde{H} \rightarrow H$ such that $[\widetilde{\eta}] \circ [\pi'] = [\pi] \circ [\eta]$ and $Z_{[\pi']}$ maps $\widetilde{\overline{s}}$ to \overline{s} .*

Furthermore, the endoscopic triple $\widetilde{\mathcal{E}}$ satisfies the following properties.

(b) *For an embedding of maximal torus $a : T \hookrightarrow G$ and an element $\kappa \in \widehat{T}^\Gamma / Z(\widehat{G})^\Gamma$, denote by $\widetilde{a} : \widetilde{T} \hookrightarrow \widetilde{G}$ the lift of a , and by $\widetilde{\kappa} \in \widehat{\widetilde{T}}^\Gamma / Z(\widehat{\widetilde{G}})^\Gamma$ the image of κ . If $\mathcal{E} \cong \mathcal{E}_{([a], \kappa)}$, then $\widetilde{\mathcal{E}} \cong \mathcal{E}_{([\widetilde{a}], \widetilde{\kappa})}$.*

(c) *We have $\Lambda(\mathcal{E}) = \Lambda(\widetilde{\mathcal{E}})$ and $Z(\widetilde{\mathcal{E}}) = Z(\mathcal{E})$.*

(d) *In the notation of (b), the map sending an embedding of a maximal torus $b : T \hookrightarrow H$ to its lift $\widetilde{b} : \widetilde{T} \hookrightarrow \widetilde{H}$ induces a bijection between $\Pi_{\mathcal{E}}^{-1}([a], \overline{\kappa})$ and $\Pi_{\widetilde{\mathcal{E}}}^{-1}([\widetilde{a}], \widetilde{\overline{\kappa}})$. Moreover, for each two embeddings b_1 and b_2 , we have $\kappa \begin{pmatrix} [b_1] \\ [b_2] \end{pmatrix} (\overline{s}) = \kappa \begin{pmatrix} [\widetilde{b}_1] \\ [\widetilde{b}_2] \end{pmatrix} (\widetilde{\overline{s}})$.*

Proof. (a) Choose $\eta \in [\eta]$, a quasi-isogeny $\widehat{\pi} : \widehat{G} \rightarrow \widehat{\widetilde{G}}$ corresponding to π , and an \mathcal{E} -compatible representative $s \in \widehat{H}^\Gamma / Z(\widehat{G})^\Gamma$ of \overline{s} . Identify \widehat{H} with $\eta(\widehat{H}) \subset \widehat{G}$ and put $\widetilde{s} := \widehat{\pi}(s) \in \widehat{\widetilde{G}} / Z(\widehat{\widetilde{G}})$.

Set $\widehat{\widehat{H}} := (\widehat{\widehat{G}})_s^0$. Then $\widehat{\pi}$ induces a quasi-isogeny $\widehat{\pi}' : \widehat{H} \rightarrow \widehat{\widehat{H}}$, and we have a canonical isomorphism $\widehat{\widehat{H}} \xrightarrow{\sim} \widehat{H} \times_{Z(\widehat{G})} Z(\widehat{\widehat{G}})$. Since both homomorphisms $\rho_H : \Gamma \rightarrow \text{Out}(\widehat{H})$ and $\rho_{\widehat{G}} : \Gamma \rightarrow \text{Aut}(Z(\widehat{\widehat{G}}))$ induce the natural Galois action on $Z(\widehat{\widehat{G}})$, their product defines a homomorphism $\rho : \Gamma \rightarrow \text{Out}(\widehat{\widehat{H}})$.

Denote by \widetilde{H} the quasi-split group over E , corresponding to the pair (\widetilde{H}, ρ) (see 1.2.2 (a)), by $\widetilde{s} \in \pi_0(Z(\widetilde{H})^\Gamma / Z(\widetilde{G})^\Gamma)$ the class of \widetilde{s} , by $[\widetilde{\eta}]$ the conjugacy class of the inclusion $\widetilde{H} \hookrightarrow \widetilde{G}$, and by $[\pi']$ the conjugacy class of quasi-isogenies $\widetilde{H} \rightarrow H$ corresponding to $\widehat{\pi}'$. Then $\widetilde{\mathcal{E}} := (\widetilde{H}, [\widetilde{\eta}], \widetilde{s})$ is an endoscopic triple for \widetilde{G} , and the pair $(\widetilde{\mathcal{E}}, [\pi'])$ satisfy the required properties. The proof of the uniqueness is similar.

(b) follow immediately from the description of $\widetilde{\mathcal{E}}$ in (a).

(c) For each $\alpha \in \Lambda(\mathcal{E})$, the corresponding element $\widehat{\alpha} \in \text{Out}(\widehat{H})$ is Γ -invariant and is induced by an element of $G^{\text{ad}} = \widetilde{G}^{\text{ad}}$. Therefore α induces a unique element $\widetilde{\alpha} \in \Lambda(\widetilde{\mathcal{E}})$ and vice versa. Also we have an equality $\pi_{\mathcal{E}} = \pi_{\widetilde{\mathcal{E}}}$ implying that $Z(\mathcal{E}) = Z(\widetilde{\mathcal{E}})$.

(d) is clear. \square

1.4. Endoscopic triples: further properties.

Let $\mathcal{E} = (H, [\eta], \overline{s})$ be an elliptic endoscopic triple for G , and $b : T \hookrightarrow H$ an embedding of a maximal torus. Put $([a], \overline{\kappa}) := \Pi_{\mathcal{E}}([b])$, and denote by $S_{[b]}$ the subset $\left\{ \kappa \left(\frac{[b_1]}{[b]} \right) (\overline{s}) \right\}_{[b_1] \in \Pi_{\mathcal{E}}^{-1}([a], \overline{\kappa})}$ of $\pi_0(\widehat{T/Z(G)})^\Gamma$.

The primary goal of this subsection is to prove the following result.

Proposition 1.4.1. *The subset $S_{[b]} \subset \pi_0(\widehat{T/Z(G)})^\Gamma$ is a subgroup.*

This proposition has the following corollary.

Corollary 1.4.2. *For each $z \in Z(\mathcal{E}, [a], \overline{\kappa})$, there exists $[b_1] \in \Pi_{\mathcal{E}}^{-1}([a], \overline{\kappa})$ such that $\kappa \left(\frac{[b_1]}{[b]} \right) (\overline{s}) = Z_{[\widehat{a}]}(z)$.*

Proof of the corollary. By definition, $Z_{[\widehat{a}]}(Z(\mathcal{E}, [a], \overline{\kappa})) = S_{([a], \overline{\kappa})}$, while $S_{([a], \overline{\kappa})}$ is the group generated by $S_{[b]}$. Since $S_{[b]}$ itself is a group, we get that $S_{([a], \overline{\kappa})} = S_{[b]}$, implying the assertion. \square

To prove of the proposition, we will show first several results of independent interest, while the proof itself will be carried out in 1.4.9.

As \mathcal{E} is elliptic, we denote $\overline{s} \in \pi_0(Z(\widehat{H})^\Gamma / Z(\widehat{G})^\Gamma) = Z(\widehat{H})^\Gamma / Z(\widehat{G})^\Gamma$ simply by s . Note that $\pi_0(\widehat{G}_s)$ is a Γ -invariant subgroup of $\text{Out}(\widehat{H})$.

Lemma 1.4.3. *Choose $\eta \in [\eta]$ and identify \widehat{H} with $\eta(\widehat{H}) \subset \widehat{G}$.*

(a) *There exists a natural isomorphism $\Lambda(\mathcal{E}) \cong \pi_0(\widehat{G}_s)^\Gamma$, and a Γ -equivariant injection $\iota : \pi_0(\widehat{G}_s) \hookrightarrow Z(\widehat{G}^{\text{sc}})$. Moreover, ι induces an isomorphism $\Lambda(\mathcal{E}) \xrightarrow{\sim} Z(\mathcal{E})$.*

(b) *If G is split, then the image of $\rho_H : \Gamma \rightarrow \text{Out}(\widehat{H})$ lies in $\pi_0(\widehat{G}_s)$, and $\Lambda(\mathcal{E})$ is canonically isomorphic to $\pi_0(\widehat{G}_s)$.*

Proof. For the proof we can replace G by G^{sc} and \mathcal{E} by the corresponding endoscopic triple, thus we can assume that $G = G^{\text{sc}}$.

(a) Using the fact that \mathcal{E} is elliptic, the first assertion follows from Remark 1.3.4. Next for each $g \in \widehat{G}_s$, choose a representative $\tilde{g} \in \widehat{G}^{\text{sc}}$ of g and a representative $\tilde{s} \in \widehat{G}^{\text{sc}}$ of s . Then the element $(\tilde{g}\tilde{s}\tilde{g}^{-1})\tilde{s}^{-1} \in Z(\widehat{G}^{\text{sc}})$ does not depend on the choices, and the map $g \mapsto (\tilde{g}\tilde{s}\tilde{g}^{-1})\tilde{s}^{-1}$ defines a homomorphism $\tilde{\iota} : \widehat{G}_s \rightarrow Z(\widehat{G}^{\text{sc}})$. Moreover, $g \in \text{Ker } \tilde{\iota}$ if and only if $\tilde{g} \in (\widehat{G}^{\text{sc}})_{\tilde{s}}$. By [St, §8], the last group is connected, therefore $\text{Ker } \tilde{\iota} = \widehat{G}_s^0$. Hence $\tilde{\iota}$ induces an embedding $\iota : \pi_0(\widehat{G}_s) \hookrightarrow Z(\widehat{G}^{\text{sc}})$, which is clearly Γ -equivariant.

Since \mathcal{E} is elliptic, the group $Z(\widehat{H/Z(G)})^\Gamma$ is finite. As $\pi_{\mathcal{E}}$ is the composition of $\iota|_{\Lambda(\mathcal{E})}$ with the embedding $Z(\widehat{G}^{\text{sc}})^\Gamma \hookrightarrow Z(\widehat{H/Z(G)})^\Gamma = \pi_0(Z(\widehat{H/Z(G)}))^\Gamma$, the last assertion follows.

(b) The first assertion follows from the definition. Since embedding $\iota : \pi_0(\widehat{G}_s) \hookrightarrow Z(\widehat{G}^{\text{sc}})$ is Γ -equivariant and Γ acts trivially on $Z(\widehat{G}^{\text{sc}})$, we conclude from (a) that that $\Lambda(\mathcal{E}) \cong \pi_0(\widehat{G}_s)^\Gamma = \pi_0(\widehat{G}_s)$, as claimed. \square

1.4.4. Action of $Z(\widehat{G}^{\text{sc}})$ on the extended Dynkin diagram $\tilde{D}_{\widehat{G}}$ of \widehat{G} .

Let \widehat{T}^{ad} and \widehat{T}^{sc} be the abstract Cartan subgroups of \widehat{G}^{ad} and \widehat{G}^{sc} , respectively, and let $C \subset X_*(\widehat{T}^{\text{ad}}) \otimes \mathbb{R}$ be the fundamental alcove. For every $\mu \in X_*(\widehat{T}^{\text{ad}})$ there exists a unique element $w_\mu \in W^{\text{aff}}$ of the affine Weyl group of \widehat{G} such that $w_\mu(C + \mu) = C$. Then the map $c_\mu : x \mapsto w_\mu(x + \mu)$ defines an affine automorphism of C , and hence an automorphism of $\tilde{D}_{\widehat{G}}$. Moreover, the map $c : \mu \mapsto c_\mu$ is a homomorphism, and $X_*(\widehat{T}^{\text{sc}}) \subset \text{Ker } c$. Thus c induces an action of $Z(\widehat{G}^{\text{sc}}) = X_*(\widehat{T}^{\text{ad}})/X_*(\widehat{T}^{\text{sc}})$ on C , hence on $\tilde{D}_{\widehat{G}}$.

Lemma 1.4.5. *Let G be a split simple group. Then there exists a bijection $[\alpha] \mapsto \mathcal{E}_{[\alpha]} = (H_{[\alpha]}, s_{[\alpha]}, [\eta]_{[\alpha]})$ between the set of $Z(\widehat{G}^{\text{sc}})$ -orbits of vertexes of $\tilde{D}_{\widehat{G}}$ and the set of isomorphism classes of split elliptic endoscopic triples for G .*

Moreover, for each vertex $\alpha \in \tilde{D}_{\widehat{G}}$, the stabilizer $Z(\widehat{G}^{\text{sc}})_\alpha$ is canonically isomorphic to $\Lambda(\mathcal{E}_{[\alpha]})$, and the order $\text{ord}(s_{[\alpha]})$ is equal to the coefficient of α in the reduced linear dependence $\sum_{\alpha \in \tilde{D}_{\widehat{G}}} n_\alpha \alpha = 0$.

Proof. The set of isomorphisms of split endoscopic triples for G is in bijection with the set of conjugacy classes of semisimple elements $s \in \widehat{G}^{\text{ad}}$ such that \widehat{G}_s^0 is semisimple, hence with the set of $W(\widehat{G})$ -orbits of elements $s \in \widehat{T}^{\text{ad}}$ such that $[X^*(\widehat{T}^{\text{ad}}) : X^*(\widehat{T}^{\text{ad}})_s] < \infty$.

Note that $X_*(\widehat{T}^{\text{ad}}) \otimes \mathbb{R} / X_*(\widehat{T}^{\text{ad}}) (= \text{Hom}(X_*(\widehat{T}^{\text{ad}}), \mathbb{R}/\mathbb{Z}))$ is naturally isomorphic to $\widehat{T}(\mathbb{C})^1 = \text{Hom}(X_*(\widehat{T}^{\text{ad}}), S^1)$. This isomorphism induces a bijection between $W(\widehat{G})$ -orbits on $\widehat{T}(\mathbb{C})^1$ and $(X_*(\widehat{T}^{\text{ad}}) \rtimes W(\widehat{G}))$ -orbits on $X_*(\widehat{T}^{\text{ad}}) \otimes \mathbb{R}$. Since C is a fundamental domain for the action of $W^{\text{aff}} = X_*(\widehat{T}^{\text{sc}}) \rtimes W(\widehat{G})$, the latter set coincides with the set of $Z(\widehat{G}^{\text{sc}})$ -orbits on C .

For each $s \in \widehat{T}(\mathbb{C})^1$ and each representative $\tilde{s} \in C \subset X_*(\widehat{T}^{\text{ad}}) \otimes \mathbb{R}$ of s , the set of roots α of \widehat{G} such that $\alpha(s) = 1$ are in bijection with the set of affine roots β of \widehat{G} such that $\beta(\tilde{s}) = 0$. Therefore $[X^*(\widehat{T}^{\text{ad}}) : X^*(\widehat{T}^{\text{ad}})_s] < \infty$ if and only if \tilde{s} is a vertex C , that is, a vertex of $\widetilde{D}_{\widehat{G}}$. The last assertion is clear. \square

The proof of the following result is done case-by-case.

Claim 1.4.6. *Let G be an absolutely simple group over E such that $(G^*)^{\text{sc}}$ is not isomorphic to SL_n . For every embedding $a : T \hookrightarrow G$ of a maximal torus, and $\bar{\kappa} \in \pi_0(\widehat{T}^\Gamma / Z(\widehat{G})^\Gamma)$ such that $([a], \bar{\kappa}) \in \text{Im } \Pi_{\mathcal{E}}$, we have*

$$[Z_{[\widehat{a}]}(Z(\mathcal{E}, [a], \bar{\kappa})) : Z_{[\widehat{a}]}(Z(\mathcal{E}))] \leq 2.$$

Proof. Replacing G by G^* , we can assume that G is quasi-split. Replacing G by G^{sc} and \mathcal{E} by the corresponding endoscopic triple, we can assume that $G = G^{\text{sc}}$.

Assume that our assertion is false, that is, $[Z_{[\widehat{a}]}(Z(\mathcal{E}, [a], \bar{\kappa})) : Z_{[\widehat{a}]}(Z(\mathcal{E}))] > 2$. Since $Z(\mathcal{E}, [a], \bar{\kappa})$ is a subgroup of $Z(\widehat{G}^{\text{sc}})^\Gamma$, we conclude that $|Z(\widehat{G}^{\text{sc}})^\Gamma| > 2$. Therefore by the classification of simple algebraic groups, we get that G (hence also \widehat{G}) is of type A , D or E_6 . Moreover, since the group $\text{Out}(\widehat{G})$ acts faithfully on $Z(\widehat{G}^{\text{sc}})$, we see case-by-case that the assumption $|Z(\widehat{G}^{\text{sc}})^\Gamma| > 2$ implies that G is split.

By our assumption, G is not of type A , therefore $|Z(\widehat{G}^{\text{sc}})| \leq 4$. Since by our assumption $[Z(\widehat{G}^{\text{sc}}) : Z(\mathcal{E})] > 2$, we get that $Z(\mathcal{E}) = 1$, hence $\Lambda(\mathcal{E}) = 1$ (see Lemma 1.4.3 (a)). It follows from Lemma 1.4.3 (b) that $\text{Im } \rho_H = 1$, thus \mathcal{E} is split. Therefore by Lemma 1.4.5, \mathcal{E} corresponds to a $Z(\widehat{G}^{\text{sc}})$ -orbit $[\alpha] \subset \widetilde{D}_{\widehat{G}}$. Moreover, since $\Lambda(\mathcal{E}) = 1$, we get that $||[\alpha]| = |Z(\widehat{G}^{\text{sc}})| > 2$.

Recall that $Z_{[\widehat{a}]}(Z(\mathcal{E}, [a], \bar{\kappa}))$ consists of images of $s = s_{[\alpha]}$ under certain homomorphisms $\kappa \left(\frac{[b_1]}{[b_2]} \right) : Z(\widehat{H}) \rightarrow \pi_0(\widehat{T}/Z(\widehat{G}))^\Gamma$. Therefore every $z \in Z_{[\widehat{a}]}(Z(\mathcal{E}, [a], \bar{\kappa}))$ satisfies $z^{\text{ord}(s_{[\alpha]})} = 1$. Since $Z_{[\widehat{a}]}(Z(\mathcal{E}, [a], \bar{\kappa})) \neq \{1\}$, we get $(\text{ord}(s_{[\alpha]}), |Z(\widehat{G}^{\text{sc}})|) \neq 1$. In particular, the orbit $[\alpha]$ is non-special, that is, consists of non-special vertexes.

Now it is easy to get a contradiction. Indeed, in the case of D_n , there are no non-special $\text{Aut}(\widehat{D}_{\widehat{G}})$ -orbits of cardinality greater than two, while in the case of E_6 , there is only one such orbit. However in this case we have $\text{ord}(s_{[\alpha]}) = 2$ and $|Z(\widehat{G}^{\text{sc}})| = 3$, contradicting the assumption that $(\text{ord}(s_{[\alpha]}), |Z(\widehat{G}^{\text{sc}})|) \neq 1$. \square

1.4.7. Restriction of scalars. (a) Let E' be a finite separable extension of E , $\Gamma' = \text{Gal}(\overline{E}/E')$, G' a reductive group over E' , and $G = R_{E'/E}G'$. Then (\widehat{G}, ρ_G) has the following description. First of all, $\widehat{G} = \prod_{\sigma \in \text{Hom}_E(E', \overline{E})} \widehat{\sigma G'}$, where $\sigma G'$ is a group over $\sigma(E')$ induced from G' by σ . Every $\tau \in \Gamma$ induces a canonical element of $\rho_{G'}(\tau, \sigma) \in \text{Isom}(\widehat{\sigma G'}, \widehat{\tau \sigma G'}) / \text{Int}(\widehat{\sigma G'})$, which coincides with $\rho_{\sigma G'}(\tau)$ if $\tau \in \text{Gal}(\overline{E}/\sigma(E'))$. Then for every $\tau \in \Gamma$ we have $\rho_G(\tau) = \prod_{\sigma} \rho_{G'}(\tau, \sigma)$.

(b) There is a canonical isomorphism $Z(\widehat{G})^{\Gamma} \xrightarrow{\sim} Z(\widehat{G'})^{\Gamma'}$.

(c) Every endoscopic triple $\mathcal{E}' = (H', [\eta'], \overline{s}')$ for G' gives rise to an endoscopic triple $\mathcal{E} = (H, [\eta], \overline{s})$ for $G = R_{E'/E}G'$, denoted by $R_{E'/E}\mathcal{E}'$, defined as follows. $H = R_{E'/E}H'$, $[\eta]$ is a product $\prod_{\sigma} [\eta'_{\sigma}]$, where $[\eta'_{\sigma}]$ is the conjugacy class of embeddings $\widehat{\sigma H'} \hookrightarrow \widehat{\sigma G'}$ induced by $[\eta']$, and \overline{s} is the preimage of \overline{s}' under the canonical isomorphism $Z(\widehat{H})^{\Gamma} / Z(\widehat{G})^{\Gamma} \xrightarrow{\sim} Z(\widehat{H'})^{\Gamma'} / Z(\widehat{G'})^{\Gamma'}$.

Lemma 1.4.8. (a) For every embedding of a maximal torus $a : T \hookrightarrow G = R_{E'/E}G'$, there exists an embedding $a' : T' \hookrightarrow G'$ of a maximal torus such that $T = R_{E'/E}T'$ and $a = R_{E'/E}a'$. Moreover, the map $[a'] \mapsto [R_{E'/E}a']$ induces a bijection between the sets of stable conjugacy classes of embeddings $T' \hookrightarrow G'$ and $T \hookrightarrow G$.

(b) For every endoscopic triple $\mathcal{E} = (H, [\eta], \overline{s})$ for G , there exists a unique endoscopic triple $\mathcal{E}' = (H', [\eta'], \overline{s}')$ for G' such that $R_{E'/E}\mathcal{E}' \cong \mathcal{E}$.

(c) In the above notation, for each $([a'], \overline{\kappa}) \in \text{Im } \Pi_{\mathcal{E}}$ the map $b' \mapsto b := R_{E'/E}b'$ induces a bijection between $\Pi_{\mathcal{E}'}^{-1}([a'], \overline{\kappa})$ and $\Pi_{\mathcal{E}}^{-1}([R_{E'/E}a'], \overline{\kappa})$. Moreover, for each two embeddings $b'_1, b'_2 : T' \hookrightarrow H'$, the isomorphism $Z(\widehat{H}/Z(\widehat{G}))^{\Gamma} \xrightarrow{\sim} Z(\widehat{H'}/Z(\widehat{G'}))^{\Gamma'}$ maps $\kappa \left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) (\overline{s})$ to $\kappa \left(\begin{bmatrix} b'_1 \\ b'_2 \end{bmatrix} \right) (\overline{s}')$.

Proof. (a) Note first that G' is a direct factor of $G_{E'}$, and denote by T' the image of the composition map $T_{E'} \xrightarrow{a} G_{E'} \rightarrow G'$. Then $R_{E'/E}T' \cong T$, and the embedding $a' : T' \hookrightarrow G'$ satisfies $R_{E'/E}a' = a$. The second assertion follows from the first one.

(b) Choose $a : T \hookrightarrow G$ and $\kappa \in \widehat{T}^{\Gamma} / Z(\widehat{G})^{\Gamma}$ such that $\mathcal{E} \cong \mathcal{E}_{([a], \kappa)}$. Let $a' : T' \hookrightarrow G'$ be the embedding as in (a), and let $\kappa' \in \widehat{T'}^{\Gamma'} / Z(\widehat{G'})^{\Gamma'}$ be the image of κ . Then $\mathcal{E}' := \mathcal{E}_{([a'], \kappa')}$ satisfies $R_{E'/E}\mathcal{E}' \cong \mathcal{E}$. The uniqueness is clear.

(c) follows immediately from (a). \square

1.4.9. Proof of Proposition 1.4.1. The proof will be carried out in two steps. First we will treat the case $G = \text{SL}_n$, and then reduce the general case to it.

Step 1: The case $G = \mathrm{SL}_n$. In this case, we will describe all the objects involved explicitly.

First of all, there exists a divisor $m|n$, a cyclic Galois extension $K \subset \overline{E}$ of E of degree m such that H is isomorphic to $(R_{K/E} \mathrm{GL}_{\frac{n}{m}})^1 = \{g \in R_{K/E} \mathrm{GL}_{\frac{n}{m}} \mid N_{K/E}(\det g) = 1\}$. Next $s \in Z(\widehat{H}) \cong (\mathbb{C}^\times)^{\mathrm{Gal}(K/E)}/\mathbb{C}^\times$ is a class $[(\mu(\sigma))_{\sigma \in \mathrm{Gal}(K/E)}]$ for a certain isomorphism $\mu : \mathrm{Gal}(K/E) \xrightarrow{\sim} \{z \in \mathbb{C}^\times \mid z^m = 1\}$.

Moreover, if we embed $H \hookrightarrow \mathrm{SL}_n$ by means of any E -linear isomorphism $K \xrightarrow{\sim} E^m$, then for every embedding of maximal torus $b : T \hookrightarrow H$, we get that $[b]_G$ is the stable conjugacy class of the composition $T \xrightarrow{b} H \hookrightarrow \mathrm{SL}_n$.

Every maximal torus $T \subset H \subset \mathrm{SL}_n$ is of the form $(\prod_{i=1}^l R_{K_i/E} \mathbb{G}_m)^1$, where $(\prod_{i=1}^l R_{K_i/E} \mathbb{G}_m)^1 = \{t_i \in \prod_{i=1}^l R_{K_i/E} \mathbb{G}_m \mid \prod_i N_{K_i/E}(t_i) = 1\}$, for certain finite extensions K_i/K with $\sum_i [K_i : K] = \frac{n}{m}$. Denote by b the inclusion $T \hookrightarrow H$. Then embeddings $b' : T \hookrightarrow H$ such that $[b'] = [b]$ (resp. $[b']_G = [b]_G$) are in canonical bijection with K -linear (resp. E -linear) algebra embeddings $\oplus_{i=1}^l K_i \hookrightarrow \mathrm{Mat}_{\frac{n}{m}}(K)$.

In particular, we get a bijection $[\iota] \mapsto b_{[\iota]}$ from the set of l -tuples $\bar{\iota} = (\iota_1, \dots, \iota_l)$ of E -algebra embeddings $\iota_i : K \hookrightarrow K_i$ to that of stable conjugacy classes $[b']$ of embeddings $T \hookrightarrow H$ such that $[b']_G = [b]_G$. Therefore both sets are principal homogeneous spaces for the action of the group $\mathrm{Gal}(K/E)^l$.

The dual torus \widehat{T} is $[\prod_{i=1}^l (\mathbb{C}^\times)^{\mathrm{Hom}_E(K_i, \overline{E})}]/\mathbb{C}^\times$, and $(\widehat{T}^\Gamma)^0$ is the image of the diagonal map $[(\mathbb{C}^\times)^l]/\mathbb{C}^\times \hookrightarrow \widehat{T}$. Also $\widehat{T/Z(G)}$ consists of $\{c_{i, \sigma_i}\}_{i, \sigma_i} \in \prod_i (\mathbb{C}^\times)^{\mathrm{Hom}_E(K_i, \overline{E})}$ such that $\prod_i \prod_{\sigma_i} c_{i, \sigma_i} = 1$.

For each stable conjugacy class $[b_{\bar{\iota}}]$ of embeddings $T \hookrightarrow H$, the corresponding embedding $Z_{[\widehat{b_{\bar{\iota}}}]} : Z(\widehat{H}) \hookrightarrow \widehat{T}$ sends $s \in Z(\widehat{H})$ to an element $[(c_{i, \sigma_i})_{i, \sigma_i}]$, given by the rule $c_{i, \sigma_i} := \mu(\sigma_i \circ \iota_i)$ (here $\sigma_i \circ \iota_i \in \mathrm{Hom}_E(K, \overline{E}) = \mathrm{Gal}(K/E)$). When $\bar{\iota}$ is replaced by $\bar{\iota} \circ \bar{\tau}$ for certain $\bar{\tau} = (\tau_1, \dots, \tau_l)$, then each c_{i, σ_i} is multiplied by $\mu(\tau_i)$. It follows that $\kappa \left(\frac{[b_{\bar{\iota} \circ \bar{\tau}}]}{[b_{\bar{\iota}}]} \right) (s)$ is the class $[(\mu(\tau_i))_{i, \sigma_i}] \in \widehat{T/Z(G)}^\Gamma$. In particular, the image of each $\kappa \left(\frac{[b_{\bar{\iota} \circ \bar{\tau}}]}{[b_{\bar{\iota}}]} \right) (s)$ in $\pi_0(\widehat{T}^\Gamma)$ is trivial. Thus (see Lemma 1.2.9) we get that $\Pi_{\mathcal{E}}([b_{\bar{\iota}}])$ is independent of $\bar{\iota}$ (hence $\Pi_{\mathcal{E}}([b_{\bar{\iota}}]) = ([a], \overline{\kappa})$ for every $\bar{\iota}$). As a result, the subset $S_{[b]}$ consists of classes of elements $[(\mu(\tau_i))_{i, \sigma_i}]$ where $\bar{\tau}$ runs through $\mathrm{Gal}(K/E)^l$. Hence $S_{[b]}$ is a group, as claimed.

Step 2: The general case. It follows from Lemma 1.3.10 (d) that the subset $S_{[b]}$ will not change if we replace G by G^{sc} , \mathcal{E} by the corresponding endoscopic triple for G^{sc} (see Lemma 1.3.10 (a)) and b by its lifting $b^{\mathrm{sc}} : T^{\mathrm{sc}} \hookrightarrow G^{\mathrm{sc}}$. Thus we are reduced to the case when G is semisimple and simply connected.

Then G is the product of its simple factors $G = \prod_i G_i$, and there exist embeddings of maximal tori $b_i : T_i \hookrightarrow G_i$ such that $T = \prod_i T_i$ and $b = \prod_i b_i$. Then $S_{[b]}$

decomposes as a product $\prod_i S_{[b_i]} \subset \prod_i \pi_0(\widehat{T_i/Z(G_i)})^\Gamma = \pi_0(\widehat{T/Z(G)})^\Gamma$. Thus it will suffice to show that each $S_{[b_i]}$ is a subgroup, thus reducing us to the case when G is simple and simply connected.

There exists a finite separable extension E' of E and an absolutely simple simply connected algebraic group G' over E' such that $G \cong R_{E'/E}G'$. Using Lemma 1.4.8, the subset $S_{[b]}$ will not change if we replace E by E' , G by G' , \mathcal{E} by \mathcal{E}' and $[b]$ by $[b']$. Thus we can assume that G is absolutely simple. Replacing G by G^* , we can assume that G is quasi-split, not isomorphic to SL_n .

For each embedding $b_1 : T \hookrightarrow H$ such that $\Pi_{\mathcal{E}}([b_1]) = \Pi_{\mathcal{E}}([b])$ and each $\alpha \in \Lambda(\mathcal{E})$ we have $\Pi_{\mathcal{E}}(\alpha([b_1])) = \Pi_{\mathcal{E}}([b])$. By Lemma 1.2.9 and Lemma 1.3.6 we have $\kappa\left(\frac{\alpha([b_1])}{[b]}\right)(\bar{s}) = \kappa\left(\frac{[b_1]}{[b]}\right)(\bar{s})Z_{[\widehat{a}]}(\pi_{\mathcal{E}}(\alpha))$. In other words, $S_{[b]}$ is invariant under the multiplication by elements from $Z_{[\widehat{a}]}(Z(\mathcal{E}))$.

On the other hand, by definition, $S_{[b]} \subset Z_{[\widehat{a}]}(Z(\mathcal{E}, [a], \bar{\kappa}))$. Since by Claim 1.4.6, we have $[Z_{[\widehat{a}]}(Z(\mathcal{E}, [a], \bar{\kappa})) : Z_{[\widehat{a}]}(Z(\mathcal{E}))] \leq 2$, we conclude that $S_{[b]}$ is equal either to $Z_{[\widehat{a}]}(Z(\mathcal{E}, [a], \bar{\kappa}))$ or to $Z_{[\widehat{a}]}(Z(\mathcal{E}))$. Hence it is a group, as claimed. \square

1.5. The case of local fields.

In this subsection we will apply the results from 1.1–1.3 to the case of endoscopic triples for a reductive group G over a local non-archimedean field E .

1.5.1. Tate–Nakayama duality. For every torus T over E , Tate–Nakayama duality provides us with a functorial isomorphism $\mathcal{D}_T : H^1(E, T) \xrightarrow{\sim} \pi_0(\widehat{T}^\Gamma)^D$ of finite abelian groups.

Kottwitz showed that for every connected reductive group G over E its cohomology group $H^1(E, G)$ has a unique structure of a finite abelian group such that for every maximal torus $T \subset G$ the natural map $H^1(E, T) \rightarrow H^1(E, G)$ is a group homomorphism (see [Ko2, Thm 1.2]). Moreover, there exists a group isomorphism $\mathcal{D}_G : H^1(E, G) \xrightarrow{\sim} \pi_0(Z(\widehat{G})^\Gamma)^D$ such that for every maximal torus T of G , the embedding $T \hookrightarrow G$ induces a commutative diagram:

$$\begin{array}{ccc} H^1(E, T) & \longrightarrow & H^1(E, G) \\ \mathcal{D}_T \downarrow & & \downarrow \mathcal{D}_G \\ \pi_0(\widehat{T}^\Gamma)^D & \longrightarrow & \pi_0(Z(\widehat{G})^\Gamma)^D. \end{array}$$

In particular, we have a canonical surjection

$$\widehat{T}^\Gamma / Z(\widehat{G})^\Gamma \rightarrow \text{Coker}[\pi_0(Z(\widehat{G})^\Gamma) \rightarrow \pi_0(\widehat{T}^\Gamma)] \xrightarrow{\sim} (\text{Ker}[H^1(E, T) \rightarrow H^1(E, G)])^D.$$

Remark 1.5.2. Borovoi [Bo] showed that for every reductive group G over E there is a functorial group isomorphism $H^1(E, G) \xrightarrow{\sim} (\pi_1(G)_\Gamma)_{\text{tor}}$, where $(\cdot)_{\text{tor}}$ means for

torsion. In particular, for every homomorphism of reductive groups $f : G_1 \rightarrow G_2$, the induced map $H^1(E, G_1) \rightarrow H^1(E, G_2)$ is a group homomorphism as well. Now the existence of Kottwitz' isomorphism \mathcal{D}_G follows from the Γ -equivariant isomorphism $\pi_1(G) \xrightarrow{\sim} X^*(Z(\widehat{G}))$.

Lemma 1.5.3. *Let $a : T \hookrightarrow G$ be an embedding of a maximal elliptic torus. Then for every inner twisting $\varphi : G \rightarrow G'$, there exists an embedding $a' : T \hookrightarrow G'$ stably conjugate to a .*

Proof. By assumption, $T/Z(G)$ is anisotropic, therefore $\widehat{T/Z(G)}^\Gamma$ is finite. Hence the canonical map $\pi_0(Z(\widehat{G^{\text{ad}}})^\Gamma) = Z(\widehat{G^{\text{ad}}})^\Gamma \hookrightarrow \widehat{T/Z(G)}^\Gamma = \pi_0(\widehat{T/Z(G)}^\Gamma)$ is injective. By duality, the canonical map $H^1(E, T/Z(G)) \rightarrow H^1(E, G^{\text{ad}})$ is surjective (see 1.5.1). This implies the assertion (see 1.1.4 (b)). \square

Lemma 1.5.4. *Assume that we are in the situation of 1.1.5 with $k = 2$. Let $\mathcal{E} = (H, [\eta], \bar{s})$ be an endoscopic triple for G , $b_1 : T_1 \hookrightarrow H$ and $b_2 : T_2 \hookrightarrow H$ are embeddings of maximal tori compatible with a_1 and a_2 , respectively. Then for every $z \in \pi_0(Z(\widehat{G})^\Gamma)$ its image $\tilde{z} := \kappa([b_1], [b_2])(z) \in \pi_0(\{(T_1 \times T_2)/Z(G)\}^\Gamma)$ (see 1.2.8) satisfies $\langle \overline{\text{inv}}((a_1, a'_1); (a_2, a'_2)), \tilde{z} \rangle = 1$.*

Proof. Recall that \tilde{z} is the image of $\mu_G(z) \in \pi_0(Z(\widehat{G^2/Z(G)})^\Gamma)$ under the map $Z_{[\widehat{a_1, a_2}]} : \pi_0(Z(\widehat{G^2/Z(G)})^\Gamma) \rightarrow \pi_0(\{(T_1 \times T_2)/Z(G)\}^\Gamma)$. Therefore by the commutative diagram of 1.5.1, we have

$$\langle \overline{\text{inv}}((a_1, a'_1); (a_2, a'_2)), \tilde{z} \rangle = \langle \Delta(\text{inv}(G, G')), \mu_G(z) \rangle.$$

Moreover, the latter expression equals $\langle \text{inv}(G, G'), \nu_G(\mu_G(z)) \rangle = \langle \text{inv}(G, G'), 0 \rangle = 1$, as claimed. \square

Notation 1.5.5. Let $\mathcal{E} = (H, [\eta], \bar{s})$ be an endoscopic triple for G , $\varphi : G \rightarrow G'$ an inner twisting, and $(a_i, a'_i, [b_i])$ be two triples consisting of stably conjugate embeddings of maximal tori $a_i : T_i \hookrightarrow G$ and $a'_i : T_i \hookrightarrow G'$, and stably conjugate classes $[b_i]$ of embeddings of maximal tori $T_i \hookrightarrow H$, compatible with a_i . To these data one associates elements $\overline{\text{inv}}((a_1, a'_1); (a_2, a'_2)) \in H^1(E, (T_1 \times T_2)/Z(G))$ (see 1.1.5) and $\kappa([b_1], [b_2])(\check{s}) \in \pi_0(\{(T_1 \times T_2)/Z(G)\}^\Gamma)$ for every representative $\check{s} \in \pi_0(Z(\widehat{H})^\Gamma)$ of \bar{s} (see 1.2.8). By Lemma 1.5.4, the pairing

$$\left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle = \left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle_{\mathcal{E}} := \langle \overline{\text{inv}}((a_1, a'_1); (a_2, a'_2)), \kappa([b_1], [b_2])(\check{s}) \rangle \in \mathbb{C}^\times$$

is independent of the choice of \check{s} .

Remark 1.5.6. This invariant is essentially the term Δ_1 of Langlands–Shelstad ([LS, (3.4)]).

Lemma 1.5.7. (a) For any three triples $(a_i, a'_i, [b_i])$, $(i = 1, 2, 3)$, we have

$$\left\langle \frac{a_1, a'_1; [b_1]}{a_3, a'_3; [b_3]} \right\rangle = \left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle \left\langle \frac{a_2, a'_2; [b_2]}{a_3, a'_3; [b_3]} \right\rangle.$$

(b) Assume that $T_1 = T_2 = T$, $a_1 = a_2 = a$ and $a'_1 = a'_2 = a'$. Then $\left\langle \frac{a, a'; [b_1]}{a, a'; [b_2]} \right\rangle = \left\langle \overline{\text{inv}}(a, a'), \kappa\left(\frac{[b_1]}{[b_2]}\right)(\bar{s}) \right\rangle$. If, moreover, $\kappa\left(\frac{[b_1]}{[b_2]}\right)(\bar{s}) = Z_{[\widehat{a}]}(z)$ for some $z \in Z(\widehat{G^{\text{ad}}})^\Gamma$, then $\left\langle \frac{a, a'; [b_1]}{a, a'; [b_2]} \right\rangle = \langle \text{inv}(G, G'), z \rangle$.

(c) Assume that $T_1 = T_2 = T$ and $b_1 = b_2 = b$. Then

$$\left\langle \frac{a_1, a'_1; [b]}{a_2, a'_2; [b]} \right\rangle = \langle \text{inv}(a_1, a_2), \bar{\kappa}_{[b]} \rangle \langle \text{inv}(a'_1, a'_2), \bar{\kappa}_{[b]} \rangle^{-1}.$$

(d) Assume that $\varphi = \text{Id}_G$. Then $\left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle = \langle \text{inv}(a_1, a'_1), \bar{\kappa}_{[b_1]} \rangle \langle \text{inv}(a_2, a'_2), \bar{\kappa}_{[b_2]} \rangle^{-1}$.

(e) Let $\pi : \tilde{G} \rightarrow G$ be a quasi-isogeny, $\tilde{\mathcal{E}}$ the endoscopic triple for \tilde{G} induced from \mathcal{E} , $\tilde{\varphi} : \tilde{G} \rightarrow \tilde{G}'$ the inner twisting induced from φ , and $\tilde{a}_i, \tilde{a}'_i$ and $[\tilde{b}_i]$ the lifts of a_i, a'_i and $[b_i]$, respectively. Then $\left\langle \frac{\tilde{a}_1, \tilde{a}'_1; [\tilde{b}_1]}{\tilde{a}_2, \tilde{a}'_2; [\tilde{b}_2]} \right\rangle_{\tilde{\mathcal{E}}} = \left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle_{\mathcal{E}}$.

(f) Let $\varphi' : G' \rightarrow G''$ be another inner twisting, and for each $i = 1, 2$, let $a''_i : T \hookrightarrow G''$ be a stable conjugate of a_i and a'_i . Then

$$\left\langle \frac{a_1, a''_1; [b_1]}{a_2, a''_2; [b_2]} \right\rangle = \left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle \left\langle \frac{a'_1, a''_1; [b_1]}{a'_2, a''_2; [b_2]} \right\rangle.$$

Proof. All assertions follow from the functoriality of the Tate–Nakayama duality 1.5.1.

(a) By Lemma 1.1.6 (c), $\left\langle \frac{a_i, a'_i; [b_i]}{a_j, a'_j; [b_j]} \right\rangle$ equals the pairing of $\overline{\text{inv}}((a_1, a'_1); (a_2, a'_2); (a_3, a'_3))$ with the image of $\kappa([b_i], [b_j])(\check{s})$ in $\pi_0((\{\prod_{i=1}^3 T_i\}/Z(G))^\Gamma)$. Thus the assertion follows from Lemma 1.2.7.

(b) The first assertion follows from Lemma 1.1.6 (d), while the second one follows from the fact that the image of $\overline{\text{inv}}(a, a')$ in $H^1(E, G^{\text{ad}})$ equals $\text{inv}(G, G')$.

(c) Since $\kappa([b], [b])$ equals $\mu_T \circ Z_{[\widehat{b}]}$, we get that

$$\left\langle \frac{a_1, a'_1; [b]}{a_2, a'_2; [b]} \right\rangle = \left\langle \widehat{\mu}_T(\overline{\text{inv}}((a_1, a'_1); (a_2, a'_2))), Z_{[\widehat{b}]}(\check{s}) \right\rangle.$$

Using Lemma 1.1.6 (d) and (e), we conclude that

$$\widehat{\mu}_T(\overline{\text{inv}}((a_1, a'_1); (a_2, a'_2))) = \text{inv}(a_1, a_2) \text{inv}(a'_1, a'_2)^{-1},$$

implying the assertion.

(d) By Lemma 1.1.6 (b), $\overline{\text{inv}}((a_1, a'_1); (a_2, a'_2))$ is the image of $(\text{inv}(a_1, a'_1), \text{inv}(a_2, a'_2))$. Since the image of $\kappa([b_1], [b_2])(\check{s})$ in $\pi_0(\widehat{T}_1^\Gamma / Z(\widehat{G})^\Gamma) \times \pi_0(\widehat{T}_2^\Gamma / Z(\widehat{G})^\Gamma)$ equals $(\overline{\kappa}_{[b_1]}, \overline{\kappa}_{[b_2]}^{-1})$, the assertion follows.

(e) By 1.1.13 (d), $\overline{\text{inv}}((a_1, a'_1); (a_2, a'_2)) \in H^1(E, (T_1 \times T_2)/Z(G))$ is the image of $\overline{\text{inv}}((\tilde{a}_1, \tilde{a}'_1); (\tilde{a}_2, \tilde{a}'_2)) \in H^1(E, (\tilde{T}_1 \times \tilde{T}_2)/Z(\tilde{G}))$. Choose a representative $\check{s} \in \pi_0(Z(\widehat{H})^\Gamma)$ of \overline{s} , and let $\tilde{s} \in \pi_0(Z(\widehat{H})^\Gamma)$ be the image of \check{s} . Then $\kappa([\tilde{b}_1], [\tilde{b}_2])(\tilde{s})$ is the image of $\kappa([b_1], [b_2])(\check{s})$, and the assertion follows.

(f) Follows from Lemma 1.1.6 (f). \square

Definition 1.5.8. Let \mathcal{E} be an endoscopic triple for G , and $([a], \overline{\kappa})$ a pair belonging to $\text{Im } \Pi_{\mathcal{E}}$. An inner twisting $\varphi : G \rightarrow G'$ is called \mathcal{E} -admissible (resp. $(\mathcal{E}, [a], \overline{\kappa})$ -admissible), if the corresponding class $\text{inv}(G, G') \in H^1(E, G^{\text{ad}}) \cong (Z(\widehat{G^{\text{ad}}})^\Gamma)^D$ is orthogonal to $Z(\mathcal{E}) \subset Z(\widehat{G^{\text{ad}}})^\Gamma$ (resp. orthogonal to $Z(\mathcal{E}, [a], \overline{\kappa}) \subset Z(\widehat{G^{\text{ad}}})^\Gamma$).

Lemma 1.5.9. (a) If φ is $(\mathcal{E}, [a], \overline{\kappa})$ -admissible, then φ is \mathcal{E} -admissible. The converse is true, if $a(T) \subset G$ is elliptic.

(b) For each $i = 1, 2$, the function $[b_i] \mapsto \left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle$ is constant on the fiber $\Pi_{\mathcal{E}}^{-1}([a_i], \overline{\kappa})$ if and only if φ is $(\mathcal{E}, [a_i], \overline{\kappa})$ -admissible.

Proof. (a) The assertion is a translation of Lemma 1.3.9 (b).

(b) We will show the assertion for $i = 1$, while the case $i = 2$ is similar. For each $[b_1], [b'_1] \in \Pi_{\mathcal{E}}^{-1}([a_1], \overline{\kappa})$, the quotient $\left\langle \frac{a_1, a'_1; [b'_1]}{a_2, a'_2; [b_2]} \right\rangle / \left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle$ equals the pairing $\left\langle \overline{\text{inv}}(a_1, a'_1), \kappa \left(\frac{[b_1]}{[b'_1]} \right) (\overline{s}) \right\rangle$ (use Lemma 1.5.7 (a) and (b)). Since by definition, elements $\kappa \left(\frac{[b_1]}{[b'_1]} \right) (\overline{s})$ run through $Z_{[\overline{a}]}(Z(\mathcal{E}, [a_1], \overline{\kappa}))$, our assertion follows from the last assertion of Lemma 1.5.7 (b). \square

Notation 1.5.10. (a) Assume that in Notation 1.5.5, φ is $(\mathcal{E}, [a_1], \overline{\kappa}_{[b_1]})$ -admissible (resp. $(\mathcal{E}, [a_2], \overline{\kappa}_{[b_2]})$ -admissible), then we will denote $\left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle$ by $\left\langle \frac{a_1, a'_1; \overline{\kappa}_{[b_1]}}{a_2, a'_2; [b_2]} \right\rangle$ (resp. $\left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; \overline{\kappa}_{[b_2]}} \right\rangle$). This notion is well defined by Lemma 1.5.9 (b).

(b) If in addition, $\varphi_X : X \rightarrow X'$ is an inner twisting induced by φ , $a_1 = a_x$ and $a'_1 = a_{x'}$ (see 1.1.1 (c) and 1.1.9 (c)), then we will denote $\left\langle \frac{a_1, a'_1; \overline{\kappa}_{[b_1]}}{a_2, a'_2; [b_2]} \right\rangle$ simply by $\left\langle \frac{x, x'; \overline{\kappa}_{[b_1]}}{a_2, a'_2; [b_2]} \right\rangle$.

1.6. Definitions of stability and equivalence.

In this subsection we will define the notions of stability and equivalence of invariant generalized functions.

1.6.1. Set up. (a) Let G be a connected reductive group over a local non-archimedean field E , ω_G a non-zero translation invariant differential form on G of the top degree, and $dg := |\omega_G|$ the corresponding Haar measure on $G(E)$. Let $\omega_{\mathcal{G}}$ a non-zero translation invariant differential form on \mathcal{G} of the top degree such that the identification $T_1(G) = \mathcal{G} = T_0(\mathcal{G})$ identifies $\omega_G|_{g=1}$ with $\omega_{\mathcal{G}}|_{x=0}$.

We also assume that \mathcal{G} contain regular semisimple elements, which hold automatically if the characteristic of E is different from two.

(b) Let (X, ω_X) be a pair consisting of a smooth algebraic variety X over E , equipped with an G^{ad} -action, and a non-vanishing G -invariant top degree differential form ω_X on X , and let $dx := |\omega_X|$ be the corresponding measure on $X(E)$. We assume that $X^{\text{sr}} \subset X$ is Zariski dense. This condition automatically implies that $X^{\text{sr}} \subset X$ is open (see Lemma 1.7.6 (a), below). In order to avoid dealing with algebraic spaces in Subsection 1.7, we assume that X is quasi-projective.

The results from this and the next subsections will be later used in two particular cases: $(X, \omega_X) = (G, \omega_G)$ and $(X, \omega_X) = (\mathcal{G}, \omega_{\mathcal{G}})$ with the actions of $\text{Int } G = G^{\text{ad}}$ and $\text{Ad } G = G^{\text{ad}}$, respectively.

(c) Let $\varphi : G \rightarrow G'$ be an inner twisting. The inner twist X' of X is smooth, and the differential form $\omega_{X'} := (\varphi_X^{-1})^*(\omega_X)$ on X' is E -rational and G'^{ad} -invariant.

We also denote by G^* the quasi-split inner twist of G , and by X^* the corresponding twist of X .

(d) Let $\mathcal{E} = (H, [\eta], \bar{s})$ be an endoscopic triple for G .

Notation 1.6.2. (a) let $C_c^\infty(X(E))$ be the space of locally constant functions on $X(E)$ with compact support, and let $\mathcal{S}(X(E))$ be the space of locally constant measures on $X(E)$ with compact support, that is, measures of the form $\phi = f dx$, where $f \in C_c^\infty(X(E))$.

Denote by $\mathcal{D}(X(E)) = (\mathcal{S}(X(E)))^{G(E)}$ the space of $G(E)$ -invariant linear functionals on $\mathcal{S}(X(E))$, which we call *(invariant) generalized functions*.

(b) Let $U \subset X(E)$ be an open and closed subset. For each $\phi = f dx \in \mathcal{S}(X(E))$, put $\phi|_U := (f|_U) dx \in \mathcal{S}(X(E))$. Moreover, if U is $G(E)$ -invariant, then for each $F \in \mathcal{D}(X(E))$, the generalized function $F|_U$ given by the formula $F|_U(\phi) := F(\phi|_U)$ belongs to $\mathcal{D}(X(E))$.

(c) For every smooth morphism $\pi : X_1 \rightarrow X_2$, the integration along fibers $\pi_!$ maps $\mathcal{S}(X_1(E))$ to $\mathcal{S}(X_2(E))$. Moreover, if π is G -equivariant, then the dual of $\pi_!$ induces a map $\pi^* : \mathcal{D}(X_2(E)) \rightarrow \mathcal{D}(X_1(E))$.

Remark 1.6.3. The map $\phi \mapsto \frac{\phi}{dx}$ identifies $\mathcal{S}(X(E))$ with $C_c^\infty(X(E))$, hence the space $\mathcal{D}(X(E))$ with the space of invariant distributions on $X(E)$. Below we list several reasons why $\mathcal{S}(X(E))$ and $\mathcal{D}(X(E))$ are more convenient to work with.

i) The space $C_c^\infty(X(E))$ is not functorial with respect to non-proper maps.

- ii) Characters of admissible representations of $G(E)$ belong to $\mathcal{D}(X(E))$.
- iii) Orbital integrals behave better (see Remark 1.6.5 below).

Notation 1.6.4. For each $x \in X^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$,

(i) fix an invariant measure dg_x on $G_x(E)$ such that the total measure of the maximal compact subgroup of $G_x(E)$ is 1, and define orbital integral $O_x \in \mathcal{D}(X(E))$ by the formula

$$O_x(\phi) := \int_{G(E)/G_x(E)} \left(\frac{\phi}{dx} \right) (g(x)) \frac{dg}{dg_x}$$

for each $\phi \in \mathcal{S}(X(E))$.

(ii) denote by $O_x^{\bar{\xi}} \in \mathcal{D}(X(E))$ the sum $\sum_{x'} \langle \text{inv}(x, x'), \bar{\xi} \rangle O_{x'}$, taken over a set of representatives $x' \in X(E)$ of conjugacy classes stably conjugate to x .

(iii) When $\bar{\xi} = 1$, we will write SO_x instead of $O_x^{\bar{\xi}}$. More generally, for each $x^* \in (X^*)^{\text{sr}}(E)$ we define $SO_{x^*} \in \mathcal{D}(X(E))$ be zero unless there exists a stable conjugate $x \in X(E)$ of x^* , in which case, $SO_{x^*} := SO_x$ (compare Corollary 1.1.11).

Remark 1.6.5. If (X, ω_X) is either (G, ω_G) or $(\mathcal{G}, \omega_{\mathcal{G}})$, then measure dx is induced by dg , and orbital integrals O_x are independent of a choice of dg .

Definition 1.6.6. (i) A measure $\phi \in \mathcal{S}(X(E))$ is called \mathcal{E} -unstable if $O_x^{\bar{\xi}}(\phi) = 0$ for all $x \in X^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$ such that $([a_x], \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$.

(ii) A generalized function $F \in \mathcal{D}(X(E))$ is called \mathcal{E} -stable if $F(\phi) = 0$ for all \mathcal{E} -unstable $\phi \in \mathcal{S}(X(E))$.

Remark 1.6.7. Denote by $\mathcal{D}^0(X(E)) \subset \mathcal{D}(X(E))$ the closure of the linear span of $\{O_x\}_{x \in X^{\text{sr}}(E)}$. Our notion of \mathcal{E} -stability (and of $(a, a'; [b])$ -equivalence below) seems to be "correct" only for generalized functions belonging to $\mathcal{D}^0(X(E))$.

However, all generalized functions considered in this paper belong to $\mathcal{D}^0(X(E))$. Indeed, if (X, ω_X) is either (G, ω_G) or $(\mathcal{G}, \omega_{\mathcal{G}})$, it follows from the results of Harish-Chandra [HC2, Thm. 3.1] (at least when the characteristic of E is zero) that $\mathcal{D}^0(X(E)) = \mathcal{D}(X(E))$ (see also Remarks 1.7.2 and 1.7.12 below).

Notation 1.6.8. Fix a triple $(a, a'; [b])$, consisting of stably conjugate embeddings of maximal tori $a : T \hookrightarrow G$, $a' : T \hookrightarrow G'$, and a stable conjugacy class $[b]$ of embeddings $T \hookrightarrow H$, compatible with a and a' . For every $\phi' \in \mathcal{S}(X'(E))$, $x \in X^{\text{sr}}(E)$ and embedding $c : G_x \hookrightarrow H$ compatible with $a_x : G_x \hookrightarrow G$, we define

$$(O_x^{[c]})_{(a, a'; [b])} := \sum_{x'} \left\langle \frac{a_x, a_{x'}; [c]}{a, a'; [b]} \right\rangle O_{x'} \in \mathcal{D}(X'(E)),$$

where the sum is taken over a set of representatives $x' \in X'(E)$ of conjugacy classes stably conjugate to x .

Remark 1.6.9. If $\varphi = \text{Id}_G$ (and $a' = a$), then $(O_x^{[c]})_{(a,a;[b])} = O_x^{\bar{\kappa}[c]}$ (by Lemma 1.5.7 (d)). In general, $(O_x^{[c]})_{(a,a';[b])}$ vanishes unless there exists a stable conjugate $x' \in X'(E)$ of x , in which case, $(O_x^{[c]})_{(a,a';[b])} = \left\langle \frac{a_x, a_{x'}; [c]}{a, a'; [b]} \right\rangle O_{x'}^{\bar{\kappa}[c]}$ (by Lemma 1.5.7 (a) and (c)).

Definition 1.6.10. Let $(a, a'; [b])$ be as in Notation 1.6.8. By Lemma 1.1.10 (b), we can choose a stably conjugate embedding $a^* : T \hookrightarrow G^*$ of a and a' .

(a) Measures $\phi \in \mathcal{S}(X(E))$ and $\phi' \in \mathcal{S}(X'(E))$ are called $(a, a'; [b])$ -indistinguishable, if for each $x^* \in (X^*)^{\text{sr}}(E)$ and each embedding $c : G_{x^*}^* \hookrightarrow H$ compatible with $a_{x^*} : G_{x^*}^* \hookrightarrow G^*$, we have

$$(1.6.1) \quad (O_{x^*}^{[c]})_{(a^*, a'; [b])}(\phi) = (O_{x^*}^{[c]})_{(a^*, a'; [b])}(\phi').$$

(b) Generalized functions $F \in \mathcal{D}(X(E))$ and $F' \in \mathcal{D}(X'(E))$ are called $(a, a'; [b])$ -equivalent if $F(\phi) = F'(\phi')$ for every pair of $(a, a'; [b])$ -indistinguishable measures $\phi \in \mathcal{S}(X(E))$ and $\phi' \in \mathcal{S}(X'(E))$.

Lemma 1.6.11. Measures $\phi \in \mathcal{S}(X(E))$ and $\phi' \in \mathcal{S}(X'(E))$ are $(a, a'; [b])$ -indistinguishable if and only if the following conditions are satisfied:

- (i) For each $x \in X^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$ such that $([a_x], \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$ and x does not have a stable conjugate element in $X'(E)$, we have $O_x^{\bar{\xi}}(\phi) = 0$.
- (ii) Condition (i) holds if x, X, G, ϕ are replaced by x', X', G', ϕ' .
- (iii) For each stable conjugate $x \in X^{\text{sr}}(E)$ and $x' \in X'^{\text{sr}}(E)$ and each $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$ such that $([a_x], \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$ we have
- (iii)' $O_x^{\bar{\xi}}(\phi) = O_{x'}^{\bar{\xi}}(\phi') = 0$, if φ is not $(\mathcal{E}, [a_x], \bar{\xi})$ -admissible, and
- (iii)'' $O_x^{\bar{\xi}}(\phi) = \left\langle \frac{x, x'; \bar{\xi}}{a, a'; [b]} \right\rangle O_{x'}^{\bar{\xi}}(\phi')$, if φ is $(\mathcal{E}, [a_x], \bar{\xi})$ -admissible.

Proof. Fix $x^* \in (X^*)^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_{x^*}^* / Z(\widehat{G})^\Gamma)$ such that $([a_{x^*}], \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$. Using Remark 1.6.9, we see that equalities (1.6.1) for all $[c] \in \Pi_{\mathcal{E}}^{-1}([a_{x^*}], \bar{\xi})$ are equivalent to equalities

- $0 = 0$, if there are no stable conjugates of x^* neither in $X(E)$ nor in $X'(E)$;
- $O_{x^*}^{\bar{\xi}}(\phi) = 0$, if there exists a stable conjugate $x \in X(E)$ of x^* but there is not such conjugate in $X'(E)$;
- $O_{x'}^{\bar{\xi}}(\phi') = 0$, if there exists a stable conjugate $x' \in X'(E)$ of x^* but there is not such conjugate in $X(E)$;
- $O_x^{\bar{\xi}}(\phi) = \left\langle \frac{a_x, a_{x'}; [c]}{a, a'; [b]} \right\rangle O_{x'}^{\bar{\xi}}(\phi')$ for all $[c] \in \Pi_{\mathcal{E}}^{-1}([a_x], \bar{\xi})$, if there exist stable conjugates $x \in X(E)$ and $x' \in X'(E)$ of x^* (use Lemma 1.5.7 (f)).

Moreover, by Lemma 1.5.9 (b), the last equalities are equivalent to the equalities $(iii)'$ and $(iii)''$. Now the assertion follows from Corollary 1.1.11. \square

Corollary 1.6.12. (a) *The notion of $(a, a'; [b])$ -equivalence is independent of the choice of a^* .*

(b) *Every two $(a, a'; [b])$ -equivalent generalized functions F and F' are \mathcal{E} -stable.*

(c) *Assume that φ is not \mathcal{E} -admissible. Then every \mathcal{E} -stable $F \in \mathcal{D}(X(E))$ and $F' \in \mathcal{D}(X'(E))$ are $(a, a'; [b])$ -equivalent.*

(d) *Assume that $a(T)$ is elliptic. If $F \in \mathcal{D}(X(E))$ and $F' \in \mathcal{D}(X'(E))$ are $(a, a'; [b])$ -equivalent, then they are $(a, a'; [b'])$ -equivalent for all $b' : T \hookrightarrow H$ such that $\Pi_{\mathcal{E}}([b']) = \Pi_{\mathcal{E}}([b])$.*

Proof. All assertions follow almost immediately from Lemma 1.6.11.

(a) is clear.

(b) By duality, we have to check that every \mathcal{E} -unstable measures $\phi \in \mathcal{S}(X(E))$ and $\phi' \in \mathcal{S}(X'(E))$ are $(a, a'; [b])$ -indistinguishable, which is clear.

(c) By duality, we have to check that every $(a, a'; [b])$ -indistinguishable $\phi \in \mathcal{S}(X(E))$ and $\phi' \in \mathcal{S}(X'(E))$ are \mathcal{E} -unstable. Hence the assertion follows from the first assertion of Lemma 1.5.9 (a).

(d) When φ is not \mathcal{E} -admissible, the assertion was proved in (c). When φ is \mathcal{E} -admissible, the assertion follows from Lemma 1.5.9. \square

Corollary 1.6.13. *Let $\pi : \tilde{G} \rightarrow G$ be a quasi-isogeny, and let $\tilde{\varphi} : \tilde{G} \rightarrow \tilde{G}', \tilde{\mathcal{E}}, (\tilde{a}, \tilde{a}'; [\tilde{b}])$ be the corresponding objects for \tilde{G} . Generalized functions $F \in \mathcal{D}(X(E))$ and $F' \in \mathcal{D}(X'(E))$ are $(\tilde{a}, \tilde{a}'; [\tilde{b}])$ -equivalent if and only if they are $(a, a'; [b])$ -equivalent.*

Proof. By duality, we have to show that measures $\phi \in \mathcal{S}(X(E))$ and $\phi' \in \mathcal{S}(X'(E))$ are $(a, a'; [b])$ -indistinguishable if and only if they are $(\tilde{a}, \tilde{a}'; [\tilde{b}])$ -indistinguishable. It follows from Lemma 1.3.10, that for each $x \in X^{\text{sr}}(E)$ and $\tilde{\xi} \in \pi_0(\widehat{\tilde{G}_x}^{\Gamma} / Z(\widehat{\tilde{G}})^{\Gamma})$ such that $([\tilde{a}_x], \tilde{\xi}) \in \text{Im } \Pi_{\tilde{\mathcal{E}}}$, there exists $\bar{\xi} \in \pi_0(\widehat{G_x}^{\Gamma} / Z(\widehat{G})^{\Gamma})$ such that $(a_x, \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$ and $\tilde{\xi}$ is the image of $\bar{\xi}$. Moreover, we have $O_x^{\tilde{\xi}} = c O_x^{\bar{\xi}}$, where $c \in \mathbb{C}^{\times}$ is such that measure $\frac{d\tilde{g}}{d\tilde{g}_x}$ on $(\tilde{G}/\tilde{G}_x)(E) = (G/G_x)(E)$ equals $c \frac{dg}{dg_x}$. Therefore the assertion follows from Lemmas 1.6.11, 1.5.7 (e), and 1.3.10 (c), (d). \square

Definition 1.6.14. Let $a : T \hookrightarrow G$ and $a' : T \hookrightarrow G'$ be stable conjugate embeddings of maximal elliptic tori, and let κ be an element of $\widehat{T}^{\Gamma} / Z(\widehat{G})^{\Gamma}$ such that $([a], \kappa) \in \text{Im } \Pi_{\mathcal{E}}$. We say that $F \in \mathcal{D}(X(E))$ and $F' \in \mathcal{D}(X'(E))$ are $(a, a'; \kappa)$ -equivalent if they are $(a, a'; [b])$ -equivalent for some or, equivalently, for all $[b] \in \Pi_{\mathcal{E}}^{-1}([a], \kappa)$ (see Corollary 1.6.12 (d)).

Lemma 1.6.15. (a) Let $\pi : X_1 \rightarrow X_2$ be a smooth G -equivariant morphism. For every \mathcal{E} -unstable $\phi \in \mathcal{S}(X_1(E))$, its push-forward $\pi_!(\phi) \in \mathcal{S}(X_2(E))$ is \mathcal{E} -unstable.

(b) Let $\varphi : G \rightarrow G'$ be an inner twisting, and $\pi' : X'_1 \rightarrow X'_2$ the corresponding inner twisting of π . For every $(a, a'; [b])$ -indistinguishable $\phi \in \mathcal{S}(X_1(E))$ and $\phi' \in \mathcal{S}(X'_1(E))$, their push-forwards $\pi_!(\phi) \in \mathcal{S}(X_1(E))$ and $\pi'_!(\phi') \in \mathcal{S}(X'_1(E))$ are $(a, a'; [b])$ -indistinguishable.

Proof. Recall that ω_{X_1} and ω_{X_2} are global nowhere vanishing sections of sheaves of top degree differential forms $\Omega_{X_1}^{\dim X_1}$ and $\Omega_{X_2}^{\dim X_2}$, respectively. Therefore $\omega_{X_1} \otimes \pi^*(\omega_{X_2})^{-1}$ is a global nowhere vanishing section of $\Omega_{X_1}^{\dim X_1} \otimes \pi^*(\Omega_{X_2}^{\dim X_2})^{-1}$, which induces a measure, denoted by $dy := \frac{dx_1}{dx_2}$, on all fibers of $\pi(E) : X_1(E) \rightarrow X_2(E)$.

For each $x \in X_2^{\text{sr}}(E)$ and $y \in \pi(E)^{-1}(x)$, we have $y \in X_1^{\text{sr}}(E)$ and $G_x = G_y$. Moreover, for all $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$, we have $O_x^{\bar{\xi}}(\pi_!(\phi)) = \int_{\pi(E)^{-1}(x)} O_y^{\bar{\xi}}(\phi) dy$. From this the assertion follows. \square

Lemma 1.6.15 has the following corollary.

Corollary 1.6.16. (a) Let $\pi : X_1 \rightarrow X_2$ be a smooth G -equivariant morphism. For every \mathcal{E} -stable $F \in \mathcal{D}(X_2(E))$, its pullback $\pi^*(F) \in \mathcal{D}(X_1(E))$ is \mathcal{E} -stable.

(b) Let $\varphi : G \rightarrow G'$ be an inner twisting, and $\pi' : X'_1 \rightarrow X'_2$ the inner twist of π . For every $(a, a'; [b])$ -equivalent $F \in \mathcal{D}(X_2(E))$ and $F' \in \mathcal{D}(X'_2(E))$, their pullbacks $\pi^*(F) \in \mathcal{D}(X_1(E))$ and $\pi'^*(F') \in \mathcal{D}(X'_1(E))$ are $(a, a'; [b])$ -equivalent.

1.7. Locally L^1 functions.

The goal of this subsection is to write down explicitly the condition for \mathcal{E} -stability and $(a, a'; [b])$ -equivalence of generalized functions coming from invariant locally L^1 functions.

Notation 1.7.1. (a) Denote by $L_{\text{loc}}^1(X(E))$ the space of $G(E)$ -invariant locally L^1 functions on $X(E)$, whose restriction to its open subset $X^{\text{sr}}(E)$ is locally constant.

(b) We have a canonical embedding $L_{\text{loc}}^1(X(E)) \hookrightarrow \mathcal{D}(X(E))$, which sends each $F \in L_{\text{loc}}^1(X(E))$ to a generalized function $\phi \mapsto \int_{X(E)} F\phi$. For simplicity of notation, we identify functions from $L_{\text{loc}}^1(X(E))$ with the corresponding generalized functions from $\mathcal{D}(X(E))$.

Remark 1.7.2. For each $F \in L_{\text{loc}}^1(X(E))$, the corresponding generalized function is contained in $\mathcal{D}^0(X(E))$.

Notation 1.7.3. For a $G(E)$ -invariant function $F : X(E) \rightarrow \mathbb{C}$ and a pair $x \in X^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$, we put $F(x, \bar{\xi}) := \sum_{x'} \langle \text{inv}(x, x'), \bar{\xi} \rangle^{-1} F(x')$, where $x' \in X(E)$ runs over a set of representatives of $G(E) \backslash [x] \subset G(E) \backslash X(E)$.

Proposition 1.7.4. *In the notation of Definition 1.6.10,*

(a) $F \in L_{loc}^1(X(E)) \subset \mathcal{D}(X(E))$ is \mathcal{E} -stable if and only if for each pair $x \in X^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$ such that $([a_x], \bar{\xi}) \notin \text{Im } \Pi_{\mathcal{E}}$, we have $F(x, \bar{\xi}) = 0$.

(b) $F \in L_{loc}^1(X(E))$ and $F' \in L_{loc}^1(X'(E))$ are $(a, a'; [b])$ -equivalent if and only if the following two conditions are satisfied:

(i) F and F' are \mathcal{E} -stable;

(ii) for all stably conjugate $x \in X^{\text{sr}}(E)$ and $x' \in X'^{\text{sr}}(E)$ and all $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$ such that $([a_x], \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$ and φ is $(\mathcal{E}, [a_x], \bar{\xi})$ -admissible, we have

$$F'(x', \bar{\xi}) = \left\langle \frac{x, x'; \bar{\xi}}{a, a'; [b]} \right\rangle F(x, \bar{\xi}).$$

After certain preparations, the proof of the proposition will be carried out in 1.7.14.

Notation 1.7.5. Denote by $\underline{\text{Tor}}$ the variety of all maximal tori in G .

Lemma 1.7.6. (a) *The subset $X^{\text{sr}} \subset X$ is open, and there exists a smooth morphism $\pi : X^{\text{sr}} \rightarrow \underline{\text{Tor}}$ such that $\pi(x) = G_x$ for each $x \in X^{\text{sr}}$.*

(b) *There exists a geometric quotient $Y = G \backslash X^{\text{sr}}$. Moreover, the canonical projection $f : X^{\text{sr}} \rightarrow Y$ is smooth, the restriction of f to each fiber of π is étale, and the induced map $f(E) : X^{\text{sr}}(E) \rightarrow Y(E)$ is a (locally) trivial fibration.*

Proof. (a) Denote by X^{reg} the set of $x \in X$ such that $\dim \mathcal{G}_x = \text{rk}_{\overline{E}}(G)$. Then X^{reg} contains X^{sr} , therefore X^{reg} is Zariski dense in X . Our first step will be to show that X^{reg} is open in X , and the map $x \mapsto \mathcal{G}_x$ gives an algebraic morphism $\bar{\pi}$ from X^{reg} to the Grassmannian $\text{Gr}_{\mathcal{G}, \text{rk}_{\overline{E}}(G)}$, classifying linear subspaces of \mathcal{G} of dimension $\text{rk}_{\overline{E}}(G)$.

Observe that the action $\mu : G \times X \rightarrow X$ induces a map $T(\mu) : T(G) \times T(X) \rightarrow T(X)$ of tangent bundles. The restriction of $T(\mu)$ to $\mathcal{G} \times X$, where $\mathcal{G} = T_1(G) \subset T(G)$, and $X \subset T(X)$ is the zero section, is a map of vector bundles $f : \mathcal{G} \times X \rightarrow T(X)$ such that for each $x \in X$, the kernel of $f_x : \mathcal{G} \rightarrow T_x(X)$ is \mathcal{G}_x . In other words, X^{reg} can be described as the set of $x \in X$ such that $\text{rk } f_x = \dim \mathcal{G} - \text{rk}_{\overline{E}}(G)$. Since X^{reg} is dense in X , we get that $\text{rk } f_x \leq \dim \mathcal{G} - \text{rk}_{\overline{E}}(G)$ for each $x \in X$, and $X^{\text{reg}} \subset X$ is open. Moreover, the restriction $\text{Ker } f|_{X^{\text{reg}}}$ is a vector subbundle of $\mathcal{G} \times X^{\text{reg}}$, therefore it gives rise to a morphism $\bar{\pi} : X^{\text{reg}} \rightarrow \text{Gr}_{\mathcal{G}, \text{rk}_{\overline{E}}(G)}$ such that $\bar{\pi}(x) = \mathcal{G}_x$.

Next consider a subset X^{rss} of X^{reg} consisting of points x such that $\mathcal{G}_x \subset \mathcal{G}$ is a Cartan subalgebra of \mathcal{G} (hence $G_x^0 \subset G$ is a maximal torus). Since we assumed that $\mathcal{G}^{\text{rss}} \neq \emptyset$, every Cartan subalgebra of \mathcal{G} has a non-zero intersection with \mathcal{G}^{rss} . Hence X^{rss} equals the set of $x \in X^{\text{reg}}$ such that $\mathcal{G}_x \cap \mathcal{G}^{\text{rss}} \neq \emptyset$. Since \mathcal{G}^{rss} is open in \mathcal{G} , we conclude that X^{rss} is open in X^{reg} , hence in X .

Note that the map $T \mapsto \mathcal{T} = \text{Lie } T$ identifies \underline{Tor} with the variety of Cartan subalgebras of \mathcal{G} , which is a locally closed subvariety of $\text{Gr}_{\mathcal{G}, \text{rk}_{\overline{E}}(G)}$. Therefore the restriction of $\overline{\pi}|_{X^{\text{rss}}}$ can be viewed as a morphism $\pi : X^{\text{rss}} \rightarrow \underline{Tor}$ such that $\pi(x) = G_x^0$ for each $x \in X^{\text{rss}}$.

We claim that π is smooth. Since both X^{rss} and \underline{Tor} are smooth, we only have to check that the differential $d\pi_x : T_x(X) \rightarrow T_{\pi(x)}(\underline{Tor})$ is surjective for each $x \in X^{\text{rss}}$. Put $T := \pi(x)$. Then $G_x^0 = T$, $G_x \subset \text{Norm}_G(T)$, $G(x) \cong G/G_x$ and $\underline{Tor} \cong G/\text{Norm}_G(T)$. Hence $\pi|_{G(x)}$ is étale, thus $d\pi_x|_{T_x(G(x))}$ is surjective, which implies the surjectivity of $d\pi_x$.

It remains to show that X^{sr} is open in X^{rss} . Fix $T \in \underline{Tor}$, and put $W = \text{Norm}_G(T)/T$. Then W acts on $\pi^{-1}(T)$, and $Z_T := \pi^{-1}(T) \cap X^{\text{sr}}$ consists of points $x \in \pi^{-1}(T)$ such that $w(x) \neq x$ for all $w \neq 1$. Hence $Z_T \subset \pi^{-1}(T)$ is open, and W acts freely on Z_T . In particular, Z_T is smooth.

Consider the natural map $\iota : (G/T) \times_W Z_T \rightarrow X^{\text{rss}} : [g, x] \mapsto g(x)$. This is a map between smooth spaces, which induces an isomorphism between tangent spaces, hence ι is étale. Since ι induces a bijection between $(G/T) \times_W Z_T$ and $X^{\text{sr}} \subset X^{\text{rss}}$, we get that $X^{\text{sr}} \cong (G/T) \times_W Z_T$ is open in X^{rss} .

(b) Since X is quasi-projective, Z_T is quasi-projective as well. Hence a quasi-projective scheme $Y := W \backslash Z_T$ is a geometric quotient $G \backslash X^{\text{sr}} = G \backslash [(G/T) \times_W Z_T]$. Moreover, the projection $f : X^{\text{sr}} \rightarrow Y$ is smooth, and $f|_{Z_T}$ is étale.

To show the last assertion, choose $x \in X^{\text{sr}}(E)$ and put $T := G_x$. Since the projection $f|_{Z_T}$ is étale, there exist open neighborhoods $U \subset Z_T(E)$ of x and $V \subset Y(E)$ of $f(x)$ such that f induces a homeomorphism $U \xrightarrow{\sim} V$. Then the map $(G/T)(E) \times U \rightarrow X^{\text{sr}}(E)$ sending $([g], u)$ to $g(u)$ induces a $G(E)$ -equivariant isomorphism between $(G/T)(E) \times U \cong (G/T)(E) \times V$ and $f(E)^{-1}(V)$. \square

Construction 1.7.7. (a) For each $T \in \underline{Tor}(E)$ and an open and compact subset $U \subset \pi^{-1}(T)(E)$, there exists a measure $\phi_U \in \mathcal{S}(X(E))$ such that $O_x(\phi_U) = 1$ for each $x \in G(E)(U)$, and $O_x(\phi_U) = 0$ otherwise.

Explicitly, for each open and compact subgroup $K \subset G(E)$, the measure $\phi_U := \frac{|dt|(K \cap T(E))}{|dg|(K)} \chi_{K(U)} dx$, where $\chi_{K(U)}$ is the characteristic function of $K(U) \subset X^{\text{sr}}(E)$ and dt is an invariant measure on $T(E)$ such that the total measure of the maximal compact subgroup of $T(E)$ is 1, satisfies the required properties.

(b) For each stable conjugates $x \in X^{\text{sr}}(E)$ and $x' \in X'^{\text{sr}}(E)$, there exists a natural isomorphism $\varphi_{x,x'}$ between $\pi^{-1}(G_x) \subset X^{\text{sr}}$ and $\pi'^{-1}(G'_{x'}) \subset X'^{\text{sr}}$.

Explicitly, choose $g \in G(E^{\text{sep}})$ such that $x' = \varphi_X(g(x))$. Then the map $X_{E^{\text{sep}}} \rightarrow X'_{E^{\text{sep}}} : y \mapsto \varphi_X(g(y))$ maps $\pi^{-1}(G_x)$ to $\pi'^{-1}(G'_{x'})$, and the corresponding morphism $\varphi_{x,x'} : \pi^{-1}(G_x) \rightarrow \pi'^{-1}(G'_{x'})$ is E -rational, independent of g and $\varphi_{x,x'}(x) = x'$.

Corollary 1.7.8. (a) Given $x \in X^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$ such that $([a_x], \bar{\xi}) \notin \text{Im } \Pi_{\mathcal{E}}$, there exists an \mathcal{E} -unstable measure $\phi \in \mathcal{S}(X(E))$ such that $O_x^{\bar{\xi}}(\phi) \neq 0$ and $O_x^{\bar{\xi}'}(\phi) = 0$ for each $\bar{\xi}' \neq \bar{\xi}$.

(b) Let $x \in X^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$ be such that $([a_x], \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$, φ is $(\mathcal{E}, [a_x], \bar{\xi})$ -admissible and there exists a stable conjugate $x' \in X'(E)$ of x . Then there exist $(a, a'; [b])$ -indistinguishable measures $\phi \in \mathcal{S}(X(E))$ and $\phi' \in \mathcal{S}(X'(E))$ such that $O_x^{\bar{\xi}}(\phi) \neq 0$, $O_{x'}^{\bar{\xi}}(\phi') \neq 0$ and $O_x^{\bar{\xi}'}(\phi) = O_{x'}^{\bar{\xi}'}(\phi') = 0$ for each $\bar{\xi}' \neq \bar{\xi}$.

Proof. (a) Let $x_1 = x, \dots, x_n \in X(E)$ be a set of representatives of conjugacy classes stably conjugate to x . Choose an open neighborhood $U \subset \pi^{-1}(G_x)$ of x , and for each $i = 1, \dots, n$ put $U_i := \varphi_{x, x_i}(U) \subset \pi^{-1}(G_{x_i})(E)$, and let $\phi_{U_i} \in \mathcal{S}(X(E))$ be as in Construction 1.7.7 (a). Then measure $\phi := \sum_i \langle \text{inv}(x, x_i), \bar{\xi} \rangle^{-1} \phi_{U_i}$ satisfies the required property.

(b) Now choose a set of representatives $x'_1, \dots, x'_n \in X'(E)$ of conjugacy classes stably conjugate to x , and put $U'_i := \varphi_{x, x'_i}(U) \subset \pi'^{-1}(G'_{x_i})(E)$. Then measures $\phi := \sum_i \langle \text{inv}(x, x_i), \bar{\xi} \rangle^{-1} \phi_{U_i}$ and $\phi' := \sum_i \left\langle \frac{x, x'_i; \bar{\xi}}{a, a'; [b]} \right\rangle^{-1} \phi_{U'_i}$ satisfy the required property (use Lemma 1.6.11). \square

Lemma 1.7.9. (a) Let $F \in \mathcal{D}(X(E))$ be of the form $F = \sum_{\bar{\xi}} c_{x, \bar{\xi}} O_x^{\bar{\xi}}$, where $x \in X^{\text{sr}}(E)$, $c_{x, \bar{\xi}} \in \mathbb{C}$ and $\bar{\xi}$ runs over $\pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$. Then F is \mathcal{E} -stable if and only if $c_{x, \bar{\xi}} = 0$ for each $\bar{\xi}$ with $([a_x], \bar{\xi}) \notin \text{Im } \Pi_{\mathcal{E}}$.

(b) Let $F \in \mathcal{D}(X(E))$ and $F' \in \mathcal{D}(X'(E))$ be of the form $F = \sum_{\bar{\xi}} c_{x, \bar{\xi}} O_x^{\bar{\xi}}$ and $F' = \sum_{\bar{\xi}} c_{x', \bar{\xi}} O_{x'}^{\bar{\xi}}$ for some stable conjugate $x \in X^{\text{sr}}(E)$ and $x' \in X'^{\text{sr}}(E)$. Then F and F' are $(a, a'; [b])$ -equivalent if and only if they satisfy the following two conditions:

- (i) F and F' are \mathcal{E} -stable;
- (ii) for each $\bar{\xi} \in \pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma) = \pi_0(\widehat{G}'_{x'}^\Gamma / Z(\widehat{G}')^\Gamma)$ such that $([a_x], \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$ and φ is $(\mathcal{E}, [a_x], \bar{\xi})$ -admissible, we have $c_{x', \bar{\xi}} = \left\langle \frac{x, x'; \bar{\xi}}{a, a'; [b]} \right\rangle c_{x, \bar{\xi}}$.

Proof. (a) The “if” assertion is clear. The “only if” assertion follows from the equality $F(\phi) = 0$ applied to measure ϕ from Corollary 1.7.8 (a).

(b) Assume that F and F' satisfy assertions (i) and (ii). Then it follows from Lemma 1.6.11 and assertion (a), that for each $(a, a'; [b])$ -indistinguishable $\phi \in \mathcal{S}(X(E))$ and $\phi' \in \mathcal{S}(X'(E))$ we have $F(\phi) = F'(\phi')$. Conversely, assume that F and F' are $(a, a'; [b])$ -equivalent. Then condition (i) was proved in Corollary 1.6.12 (b) and condition (ii) follows from the equality $F(\phi) = F'(\phi')$ applied to measures ϕ and ϕ' from Corollary 1.7.8 (b). \square

The following result is clear.

Lemma 1.7.10. *Let $f : Z \rightarrow Y$ be a morphism of smooth algebraic varieties over E such that the induced map $f(E) : Z(E) \rightarrow Y(E)$ is a locally trivial fibration. Fix a measure μ on $Y(E)$, and let $U_1 \supset U_2 \supset \dots$ be a basis of open and compact neighborhoods of $y \in Y(E)$. Then for every locally constant function F on $Z(E)$ and every $\phi \in \mathcal{S}(Z(E))$, the sequence $\frac{1}{\mu(U_i)}F|_{f^{-1}(U_i)}(\phi)$ stabilizes.*

Notation 1.7.11. For each $x \in X^{\text{sr}}(E)$ and $F \in L_{\text{loc}}^1(X(E))$, denote by $F_x \in \mathcal{D}(X(E))$ the generalized function $\phi \mapsto SO_x(F\phi)$. For each $x^* \in (X^*)^{\text{sr}}(E)$, $F \in L_{\text{loc}}^1(X(E))$ and $F' \in L_{\text{loc}}^1(X'(E))$, we denote by $F_{x^*} \in \mathcal{D}(X(E))$ and $F'_{x^*} \in \mathcal{D}(X'(E))$ the generalized functions $\phi \mapsto SO_{x^*}(F\phi)$ and $\phi' \mapsto SO_{x^*}(F'\phi')$, respectively (see Notation 1.6.4 (iii)).

Remark 1.7.12. Clearly, F_x and F_{x^*} belong to $\mathcal{D}^0(X(E))$.

Claim 1.7.13. (a) $F \in L_{\text{loc}}^1(X(E))$ is \mathcal{E} -stable if and only if $F_x \in \mathcal{D}(X(E))$ is \mathcal{E} -stable for all $x \in X^{\text{sr}}(E)$.

(b) $F \in L_{\text{loc}}^1(X(E))$ and $F' \in L_{\text{loc}}^1(X'(E))$ are $(a, a'; [b])$ -equivalent if and only if $F_{x^*} \in \mathcal{D}(X(E))$ and $F'_{x^*} \in \mathcal{D}(X'(E))$ are $(a, a'; [b])$ -equivalent for all $x^* \in (G^*)^{\text{sr}}(E)$.

Proof. (a) Since X^{sr} is Zariski dense in X , the complement $X(E) \setminus X^{\text{sr}}(E)$ is nowhere dense. As $F \in L_{\text{loc}}^1(X(E))$, we get that F is \mathcal{E} -stable if and only if the restriction $F|_{X^{\text{sr}}(E)}$ is \mathcal{E} -stable.

Consider the map $f : X^{\text{sr}} \rightarrow Y$ from Lemma 1.7.6. Then for each $x \in X^{\text{sr}}(E)$ and $\phi \in \mathcal{S}(X(E))$, the value $F_x(\phi)$ is the limit of the stabilizing sequence $\frac{1}{\mu(U_i)}F|_{f^{-1}(U_i)}(\phi)$, where U_i is any basis of open and compact neighborhoods of $f(x) \in Y(E)$ (use Lemma 1.7.10). From this the assertion follows.

Indeed, if $F|_{X^{\text{sr}}(E)}$ is \mathcal{E} -stable, then each $F|_{f^{-1}(U_i)}$ is \mathcal{E} -stable. In particular, for every \mathcal{E} -unstable ϕ , we have $F|_{f^{-1}(U_i)}(\phi) = 0$ for each i , hence $F_x(\phi) = 0$. Conversely, assume that each F_x is \mathcal{E} -stable, and pick an \mathcal{E} -unstable ϕ . Then there exists an open disjoint covering $\{U_\alpha\}_\alpha$ of $f(\text{Supp } \phi)$ such that each $F|_{f^{-1}(U_\alpha)}(\phi) = 0$ for each α , hence $F(\phi) = \sum_\alpha F|_{f^{-1}(U_\alpha)}(\phi) = 0$. This shows that F is \mathcal{E} -stable.

(b) As in (a), F and F' are $(a, a'; [b])$ -equivalent if and only if $F|_{X^{\text{sr}}(E)}$ and $F'|_{X'^{\text{sr}}(E)}$ are $(a, a'; [b])$ -equivalent. Next since G acts trivially on Y , we get identifications $Y' = Y^* = Y$, and the projection $f : X^{\text{sr}} \rightarrow Y$ induce maps $f' : X'^{\text{sr}} \rightarrow Y$ and $f^* : (X^*)^{\text{sr}} \rightarrow Y$. Moreover, by Corollary 1.1.11, both $\text{Im } f(E)$ and $\text{Im } f'(E)$ are contained in $\text{Im } f^*(E)$. Now the assertion follows from Lemma 1.7.10 by exactly the same arguments as (a). \square

1.7.14. *Proof of Proposition 1.7.4.* (a) For each $x \in X^{\text{sr}}(E)$, we denote by N_x the cardinality of $\pi_0(\widehat{G}_x^\Gamma / Z(\widehat{G})^\Gamma)$. By Claim 1.7.13 (a), F is \mathcal{E} -stable if and only if each

F_x is stable. Since $F_x = \frac{1}{N_x} \sum_{\bar{\xi}} F(x, \xi) O_x^{\bar{\xi}}$, the assertion then is just a reformulation of Lemma 1.7.9 (a).

(b) The proof of (b) is similar. By Claim 1.7.13 (b), $F \in L_{loc}^1(X(E))$ and $F' \in L_{loc}^1(X'(E))$ are $(a, a'; [b])$ -equivalent if and only if each $F_{x^*} \in \mathcal{D}(X(E))$ and $F'_{x'^*} \in \mathcal{D}(X'(E))$ are $(a, a'; [b])$ -equivalent. By definition this means that each F_x and $F'_{x'}$ are \mathcal{E} -stable, and for every stable conjugate $x \in X^{\text{sr}}(E)$ and $x' \in X'^{\text{sr}}(E)$, F_x and $F'_{x'}$ are $(a, a'; [b])$ -equivalent. But by Claim 1.7.13 (a) and Lemma 1.7.9 (b), these conditions are equivalent to conditions (i) and (ii), respectively. \square

Corollary 1.7.15. (a) Let $\pi : X_1 \rightarrow X_2$ be a smooth G -equivariant morphism of varieties as in 1.6.1 (b) and $U \subset X_1(E)$ an open $G(E)$ -invariant subset. For each $F \in L_{loc}^1(X_2(E))$ such that $\pi^*(F)|_U$ is \mathcal{E} -stable, the restriction $F|_{\pi(U)}$ is \mathcal{E} -stable.

(b) Let $\pi' : X'_1 \rightarrow X'_2$ be the inner twisting of π and $U' \subset X'_1(E)$ an open $G'(E)$ -invariant subset. For each $F' \in L_{loc}^1(X'_2(E))$ such that $\pi^*(F)|_U$ and $\pi'^*(F')|_{U'}$ are $(a, a'; [b])$ -equivalent, the restrictions $F|_{\pi(U)}$ and $F'|_{\pi'(U')}$ are $(a, a'; [b])$ -equivalent.

Proof. (a) Since π is smooth and G -equivariant, the subset $\pi(U) \subset X_2(E)$ is open and $G(E)$ -invariant. Thus $F|_{\pi(U)}$ is defined and belongs to $L_{loc}^1(X_2(E))$. Let $(x, \bar{\xi})$ be as in Proposition 1.7.4 (a). If $[x] \cap \pi(U) = \emptyset$, then $F|_{\pi(U)}$ vanishes on $[x]$, hence $F|_{\pi(U)}(x, \bar{\xi}) = 0$. Assume now that $[x] \cap \pi(U) \neq \emptyset$. Replacing x by a stable conjugate, we can assume that $x = \pi(x_1)$ for some $x_1 \in X_1(E)$. Then $x_1 \in X_1^{\text{sr}}(E)$, $G_{x_1} = G_x$, $a_{x_1} = a_x$, and π induces a $G(E)$ - and Γ -equivariant isomorphism $[x_1] \xrightarrow{\sim} [x]$. Hence $F|_{\pi(U)}(x, \bar{\xi}) = \pi^*(F)|_U(x_1, \bar{\xi}) = 0$. Thus the assertion follows from Proposition 1.7.4 (a).

(b) Follows from Proposition 1.7.4 (b) by exactly the same arguments as (a). \square

1.8. Quasi-logarithm maps.

Starting from 1.8.5, E will be a local non-archimedean field, \mathcal{O} the ring of integers of E , \mathfrak{m} the maximal ideal of \mathcal{O} , \mathbb{F}_q the residue field of E , p the characteristic of \mathbb{F}_q , and G a reductive group over E split over E^{nr} .

Definition 1.8.1. Let G be an algebraic group over a field k . By a *quasi-logarithm* we call a G^{ad} -equivariant algebraic morphism $\Phi : G \rightarrow \mathcal{G}$ such that $\Phi(1) = 0$ and $d\Phi_1 : \mathcal{G} = T_1(G) \rightarrow \mathcal{G}$ is the identity map.

Example 1.8.2. Let $\rho : G \rightarrow \text{Aut } V$ be a representation such that the corresponding G -invariant pairing $\langle a, b \rangle_\rho := \text{Tr}(\rho(a)\rho(b))$ on \mathcal{G} is non-degenerate. Denote by $\text{pr}_\rho : \text{End } V \rightarrow \mathcal{G}$ be the projection given by the rule $\text{Tr}(\text{pr}_\rho(A)\rho(b)) = \text{Tr}(A\rho(b))$ for each $b \in \mathcal{G}$. Then the map $\Phi_\rho : g \mapsto \text{pr}_\rho(\rho(g) - \text{Id}_V)$ is a quasi-logarithm $G \rightarrow \mathcal{G}$.

Lemma 1.8.3. Let $\Phi : G \rightarrow \mathcal{G}$ be a quasi-logarithm map.

- (a) For every Borel subgroup B of G , we have $\Phi(B) \subset \text{Lie } B$;
 (b) If a Cartan subgroup of G is a maximal torus, then Φ induces a quasi-logarithm map $G_{\text{red}} := G/R_u(G) \rightarrow \mathcal{G}_{\text{red}}$

Proof. Let $T \subset B$ be a maximal torus, and $C := \text{Cent}_G(T)$ the corresponding Cartan subgroup.

(a) Since Φ is G^{ad} -equivariant, $\Phi(C)$ is contained in the set of fixed points of $\text{Ad } T$ in \mathcal{G} , that is, $\Phi(C) \subset \text{Lie } C$. Therefore

$$\Phi(\text{Int } B(C)) = \text{Ad } B(\Phi(T)) \subset \text{Ad } B(\text{Lie } C) \subset \text{Lie } B.$$

Since C is a Cartan subgroup of B , $\text{Int } B(C) \subset B$ is Zariski dense, hence $\Phi(B)$ is contained in $\text{Lie } B$.

(b) We have to show that for each $g \in G$ and $u \in R_u(G)$, we have

$$(1.8.1) \quad \Phi(gu) - \Phi(g) \in \text{Lie } R_u(G).$$

Since we assumed that T is a Cartan subgroup, $\text{Int } G(T)$ is Zariski dense in G . Therefore it is enough to check the equality (1.8.1) only for $g \in \text{Int } G(T)$, hence (since Φ is G^{ad} -equivariant and $R_u(G)$ is normal in G), only for $g = t \in T$. Consider subgroup $H = TR_u(G) \subset G$. Then T is a Cartan subgroup of H , hence $\text{Int } H(T)$ is Zariski dense in H . Since $H = T \ltimes R_u(G)$, it will therefore suffice to check (1.8.1) under the additional assumption that $tu = vt v^{-1}$ for certain $v \in R_u(G)$. In this case,

$$\Phi(tu) - \Phi(t) = (\text{Ad } v - \text{Id})(\Phi(t)) \in (\text{Ad } v - \text{Id})(\mathcal{T}) \subset \text{Lie } R_u(G),$$

as claimed. \square

Remark 1.8.4. If we do not assume that Cartan subgroup of G is a maximal torus, then the assertion (b) of the lemma is obviously false. For example, it is false for abelian groups.

From now on, G is a reductive group over a local non-archimedean field E , which is split over E^{nr} .

1.8.5. Bruhat-Tits building. (a) Denote by $\mathcal{B}(G)$ the (non-reduced) Bruhat-Tits building of G . For every point $x \in \mathcal{B}(G)$, we denote by $G_x \subset G(E)$ (resp. $\mathcal{G}_x \subset \mathcal{G}$) be corresponding parahoric subgroup (resp. subalgebra), and let $G_{x^+} \subset G_x$ (resp. $\mathcal{G}_{x^+} \subset \mathcal{G}_x$) be the pro-unipotent (resp. pro-nilpotent) radical of G_x (resp. of \mathcal{G}_x) (compare [MP1]).

(b) For each $x \in \mathcal{B}(G)$, denote by \underline{G}_x the canonical smooth connected group scheme over \mathcal{O} whose generic fiber is G and $\underline{G}_x(\mathcal{O}) = G_x$, and let \overline{G}_x be the special fiber of \underline{G}_x . Then \overline{G}_x is a connected group over \mathbb{F}_q , whose Cartan subgroup is a maximal torus. (Here we use the assumption that G splits over E^{nr}).

(c) For each $x \in \mathcal{B}(G)$, denote by L_x the quotient $(\overline{G}_x)_{\text{red}} = \overline{G}_x/R_u(\overline{G}_x)$. We have canonical identifications $L_x(\mathbb{F}_q) = G_x/G_{x+}$ and $\mathcal{L}_x := \text{Lie } L_x = \mathcal{G}_x/\mathcal{G}_{x+}$. For every $g \in G_x$ and $a \in \mathcal{G}_x$, we put $\overline{g} := gG_{x+} \in L_x(\mathbb{F}_q)$ and $\overline{a} := a + \mathcal{G}_{x+} \in \mathcal{L}_x(\mathbb{F}_q)$.

(d) If $G = T$ is a torus, then the group scheme \underline{T}_x is independent of $x \in \mathcal{B}(T)$, and coincides with the canonical \mathcal{O} -structure $T_{\mathcal{O}}$ of T . We denote by \overline{T} the special fiber of $T_{\mathcal{O}}$, and will write $T(\mathcal{O})$ instead of $T_{\mathcal{O}}(\mathcal{O})$.

Notation 1.8.6. (a) We call an invariant pairing $\langle \cdot, \cdot \rangle$ on \mathcal{G} *non-degenerate at $x \in \mathcal{B}(G)$* , if it is non-degenerate over E and the dual lattice

$$(\mathcal{G}_x)^\perp := \{x \in \mathcal{G}(E) \mid \langle x, y \rangle \in \mathfrak{m} \text{ for each } y \in \mathcal{G}_x\}$$

equals \mathcal{G}_{x+} .

(b) We call a quasi-logarithm $\Phi : G \rightarrow \mathcal{G}$ *defined at $x \in \mathcal{B}(G)$* , if Φ extends to the morphism $\Phi_x : \underline{G}_x \rightarrow \mathcal{G}_x$ of schemes over \mathcal{O} .

Lemma 1.8.7. (a) *If an invariant pairing $\langle \cdot, \cdot \rangle$ on \mathcal{G} is non-degenerate at x for some $x \in \mathcal{B}(G)$, then it is non-degenerate at x for all $x \in \mathcal{B}(G)$. In this case, $\langle \cdot, \cdot \rangle$ defines an invariant non-degenerate pairing $\langle \cdot, \cdot \rangle_x$ on \mathcal{L}_x for all $x \in \mathcal{B}(G)$.*

(b) *If a quasi-logarithm $\Phi : G \rightarrow \mathcal{G}$ over E is defined at x for some $x \in \mathcal{B}(G)$, then it is defined at x for all $x \in \mathcal{B}(G)$. In this case, Φ gives rise to a quasi-logarithm map $\overline{\Phi}_x : L_x \rightarrow \mathcal{L}_x$ for all $x \in \mathcal{B}(G)$.*

Proof. (a) Assume that $(\mathcal{G}_x)^\perp = \mathcal{G}_{x+}$. Since $\mathfrak{m}\mathcal{G}_x \subset \mathcal{G}_{x+}$, we get that $\langle a, b \rangle \in \mathcal{O}$ for every $a, b \in \mathcal{G}_x$, and the pairing on \mathcal{L}_x given by the formula $\langle \overline{a}, \overline{b} \rangle_x := \langle a, b \rangle \in \mathbb{F}_q$ for every $a, b \in \mathcal{G}_x$ is well-defined, L_x -invariant and non-degenerate.

It remains to show that for every $x, y \in \mathcal{B}(G)$, the equalities $(\mathcal{G}_x)^\perp = \mathcal{G}_{x+}$ and $(\mathcal{G}_y)^\perp = \mathcal{G}_{y+}$ are equivalent. For this we can extend scalars to E^{nr} . Also we can assume that \mathcal{G}_y is an Iwahori subalgebra of \mathcal{G}_x .

Assume first that $(\mathcal{G}_x)^\perp = \mathcal{G}_{x+}$. Since $\mathcal{G}_{x+} \subset \mathcal{G}_y \subset \mathcal{G}_x$, we get that $\mathcal{G}_{x+} \subset (\mathcal{G}_y)^\perp \subset \mathcal{G}_x$. Moreover, $(\mathcal{G}_y)^\perp/\mathcal{G}_{x+} \subset \mathcal{L}_x$ is the orthogonal complement of the Borel subalgebra $\mathcal{G}_y/\mathcal{G}_{x+} \subset \mathcal{L}_x$ with respect to the non-degenerate pairing $\langle \cdot, \cdot \rangle_x$. Hence $(\mathcal{G}_y)^\perp/\mathcal{G}_{x+}$ is the nilpotent radical of $\mathcal{G}_y/\mathcal{G}_{x+}$, thus $(\mathcal{G}_y)^\perp = \mathcal{G}_{y+}$. Conversely, assume that $(\mathcal{G}_y)^\perp = \mathcal{G}_{y+}$. Since $\mathcal{G}_x = \cup_{g \in G_x} \text{Ad } g(\mathcal{G}_y)$ and $\mathcal{G}_{x+} = \cap_{g \in G_x} \text{Ad } g(\mathcal{G}_{y+})$, we get that $\mathcal{G}_{x+} = (\mathcal{G}_x)^\perp$, as claimed.

(b) The strategy will be similar to that of (a). Assume that Φ extends to a morphism $\Phi_x : \underline{G}_x \rightarrow \mathcal{G}_x$. Then the special fiber of Φ_x is a quasi-logarithm $\overline{G}_x \rightarrow \overline{\mathcal{G}}_x$. Since $L_x = (\overline{G}_x)_{\text{red}}$, the existence $\overline{\Phi}_x$ follows from Lemma 1.8.3 (b) and the observation of 1.8.5 (b).

It remains to show that for every $x, y \in \mathcal{B}(G)$, the existence of Φ_x is equivalent to that of Φ_y . Notice first that the existence of Φ_x is equivalent to the fact that $\Phi(\underline{G}_x(\mathcal{O}_{E^{\text{nr}}})) \subset \mathcal{G}_x \otimes_{\mathcal{O}} \mathcal{O}_{E^{\text{nr}}}$ (see [BT, Prop. 1.7.6]). Thus we can extend scalars to

E^{nr} , and we are required to check that the inclusions $\Phi(G_x) \subset \mathcal{G}_x$ and $\Phi(G_y) \subset \mathcal{G}_y$ are equivalent. Also we can assume that G_y is an Iwahori subgroup of G_x .

Assume that $\Phi(G_x) \subset \mathcal{G}_x$. As we have shown, Φ induces a quasi-logarithm $\overline{\Phi}_x : L_x \rightarrow \mathcal{L}_x$. By Lemma 1.8.3 (a), we get $\overline{\Phi}_x(G_y/G_{x+}) \subset \mathcal{G}_y/\mathcal{G}_{x+}$, thus $\Phi(G_y) \subset \mathcal{G}_y$. Conversely, assume that $\Phi(G_y) \subset \mathcal{G}_y$. As Φ is G^{ad} -equivariant, the inclusion $\Phi(G_x) \subset \mathcal{G}_x$ follows from equalities $G_x = \cup_{g \in G_x} gG_yg^{-1}$ and $\mathcal{G}_x = \cup_{g \in G_x} \text{Ad } g(\mathcal{G}_y)$. \square

Lemma 1.8.7 allows us to give the following definition.

Definition 1.8.8. (a) We call an invariant pairing $\langle \cdot, \cdot \rangle$ on \mathcal{G} *non-degenerate over \mathcal{O}* , if it is non-degenerate at x for some (or, equivalently, for all) $x \in \mathcal{B}(G)$.

(b) We call a quasi-logarithm $\Phi : G \rightarrow \mathcal{G}$ *defined over \mathcal{O}* if it is defined at x for some (or, equivalently, for all) $x \in \mathcal{B}(G)$.

Lemma 1.8.9. (a) Let $\varphi : G \rightarrow G'$ be an inner twisting defined over E^{nr} . Every non-degenerate over \mathcal{O} invariant pairing $\langle \cdot, \cdot \rangle$ on \mathcal{G} gives rise to the corresponding pairing $\langle \cdot, \cdot \rangle'$ on \mathcal{G}' . Every quasi-logarithm $\Phi : G \rightarrow \mathcal{G}$ defined over \mathcal{O} gives rise to the corresponding quasi-logarithm $\Phi' : G' \rightarrow \mathcal{G}'$.

(b) If the pairing $\langle \cdot, \cdot \rangle_\rho$ on \mathcal{G} corresponding to a representation $\rho : G \rightarrow \text{Aut}(V)$ is non-degenerate over \mathcal{O} , then the corresponding quasi-logarithm $\Phi_\rho : G \rightarrow \mathcal{G}$ is defined over \mathcal{O} .

Proof. (a) Recall that φ induces isomorphisms $G_{E^{\text{nr}}} \xrightarrow{\sim} G'_{E^{\text{nr}}}$ and $\mathcal{G}_{E^{\text{nr}}} \xrightarrow{\sim} \mathcal{G}'_{E^{\text{nr}}}$. Hence Φ and $\langle \cdot, \cdot \rangle$ give rise to a quasi-logarithm $\Phi' : G'_{E^{\text{nr}}} \rightarrow \mathcal{G}'_{E^{\text{nr}}}$ and a pairing $\langle \cdot, \cdot \rangle' : \mathcal{G}_{E^{\text{nr}}} \times \mathcal{G}_{E^{\text{nr}}} \rightarrow E^{\text{nr}}$, respectively. Furthermore, since the twisting φ is inner, while Φ and $\langle \cdot, \cdot \rangle$ are G^{ad} -equivariant, the quasi-logarithm Φ' and the inner twisting $\langle \cdot, \cdot \rangle'$ are defined over E . Finally, to show that Φ' and $\langle \cdot, \cdot \rangle'$ are defined over \mathcal{O} , we can extend scalars to E^{nr} . Then the assertion follows from the corresponding assertion for Φ and $\langle \cdot, \cdot \rangle$.

(b) For the proof we can replace E by its finite unramified extension, so that G is split over E . Fix a hyperspecial vertex $x \in \mathcal{B}(G)$, and we have to show that $\Phi_\rho(G_x) \subset \mathcal{G}_x$. As $(\mathcal{G}_x)^\perp = \mathcal{G}_{x+}$, it is enough to show that $\text{Tr}(\rho(g)\rho(a)) \in \mathfrak{m}$ for every $g \in G_x$ and $a \in \mathcal{G}_{x+} = \mathfrak{m}\mathcal{G}_x$. Choose any $\rho(G_x)$ -invariant \mathcal{O} -lattice $V_x \subset V$. Then V_x is $\rho(\mathcal{G}_x)$ -invariant, hence $\rho(g)\rho(a)(V_x) \subset \mathfrak{m}V_x$. Thus $\text{Tr}(\rho(g)\rho(a)) \in \mathfrak{m}$, as claimed. \square

Definition 1.8.10. We say that the group G over E *satisfies property (vg)*, if G^{sc} admits a quasi-logarithm map $G^{\text{sc}} \rightarrow \mathcal{G}^{\text{sc}}$ defined over \mathcal{O} , \mathcal{G}^{sc} admits an invariant pairing non-degenerate over \mathcal{O} , and p does not divide the order of $Z(G^{\text{sc}})$.

Remark 1.8.11. By Lemma 1.8.9 (a), for every inner twisting $\varphi : G \rightarrow G'$, the group G satisfies property (vg) if and only if G' satisfies property (vg).

Lemma 1.8.12. *Write $(G^*)^{\text{sc}}$ in the form $\prod_i R_{E_i/E} H_i$, where each H_i is a quasi-split absolutely simple over a finite unramified extension E_i of E . Then G satisfies property (vg), if the following conditions are satisfied:*

- (i) p is good for each H_i in the sense of [SS, I, §4];
- (ii) p does not divide the order of each $Z(H_i)$;
- (iii) p does not divide $[E_i[H_i] : E_i]$, where $E_i[H_i]$ is the splitting field of H_i .

Proof. Assume that p satisfies assumptions (i) – (iii) of the lemma. By Lemma 1.8.9 (a), we may replace G by $(G^*)^{\text{sc}}$ hence by each $R_{E_i/E} H_i$, thus assuming that G is quasi-split simple and simply connected. By Lemma 1.8.9 (b), it will suffice to construct a representation ρ of G such that the corresponding pairing $\langle \cdot, \cdot \rangle_\rho$ on \mathcal{G} is non-degenerate over \mathcal{O} . We will construct such a ρ in three steps.

Assume first that $G = H_i$ is split. In this case, take ρ be the standard representation, if G is classical, and the adjoint representation, if G is exceptional. Then assumptions (i) and (ii) imply (as in [SS, I, Lem. 5.3]) that the pairing $\langle \cdot, \cdot \rangle_\rho$ is non-degenerate at every hyperspecial vertex of $\mathcal{B}(G)$, hence non-degenerate over \mathcal{O} .

Next we assume that $G = H_i$ is absolutely simple, set $E' := E[G]$, and put $G' := G_{E'}$. Then by the proven above, there exists a representation $\rho' : G' \rightarrow \text{Aut}_{E'}(V)$ such that the pairing $\langle \cdot, \cdot \rangle_{\rho'}$ on \mathcal{G}' is non-degenerate over $\mathcal{O}_{E'}$. Take ρ be the restriction of $R_{E'/E} \rho' : R_{E'/E} G' \rightarrow \text{Aut}_E(V)$ to G . Extending scalars to E' , we get $\rho_{E'} \cong (\rho')^{[E':E]}$, hence $\langle \cdot, \cdot \rangle_{\rho_{E'}} = [E' : E] \langle \cdot, \cdot \rangle_{\rho'}$. Therefore by assumption (iii), $\langle \cdot, \cdot \rangle_{\rho_{E'}}$ is non-degenerate over $\mathcal{O}_{E'}$, thus $\langle \cdot, \cdot \rangle_\rho$ is non-degenerate over \mathcal{O} .

In the general case, choose a representation $\rho' : H_i \rightarrow \text{Aut}_{E_i}(V)$ such that $\langle \cdot, \cdot \rangle_{\rho'}$ is non-degenerate over \mathcal{O}_{E_i} . Then the representation $\rho := R_{E_i/E} \rho' : G \rightarrow \text{Aut}_E(V)$ satisfies the required property. \square

Remark 1.8.13. The name (vg) was chosen to indicate the fact that it is closely related to the notion of a very good prime. (Recall that p is called very good for H_i , if it satisfies properties (i) and (ii) of Lemma 1.8.12.)

Notation 1.8.14. (a) For an algebraic group H , we denote by $\mathcal{U}(H) \subset H$ and $\mathcal{N}(\mathcal{H}) \subset \mathcal{H}$ the subvarieties of unipotent elements of H and of nilpotent elements of \mathcal{H} , respectively.

(b) For every $x \in \mathcal{B}(G)$, we denote by $G_{x,\text{tu}} \subset G_x$ and $\mathcal{G}_{x,\text{tn}} \subset \mathcal{G}_x$ the preimages of $\mathcal{U}(L_x)(\mathbb{F}_q) \subset L_x(\mathbb{F}_q)$ and $\mathcal{N}(\mathcal{L}_x)(\mathbb{F}_q) \subset \mathcal{L}_x$, respectively.

(c) Put $G(E)_{\text{tu}} := \cup_{x \in \mathcal{B}(G)} G_{x,\text{tu}}$ and $\mathcal{G}(E)_{\text{tn}} := \cup_{x \in \mathcal{B}(G)} \mathcal{G}_{x,\text{tn}}$.

Lemma 1.8.15. *For every $x \in \mathcal{B}(G)$,*

(a) $G_{x,\text{tu}} = \cup_y G_{y^+}$ and $\mathcal{G}_{x,\text{tn}} = \cup_y \mathcal{G}_{y^+}$, where y runs over the union of alcoves in $\mathcal{B}(G)$, whose closures contain x .

(b) $G_{x,\text{tu}} = G_x \cap G(E)_{\text{tu}} \subset G(E)$ and $\mathcal{G}_{x,\text{tn}} = \mathcal{G}_x \cap \mathcal{G}(E)_{\text{tn}} \subset \mathcal{G}(E)$.

Proof. (a) is clear.

(b) The first assertion follows from the equality $G_{x,\text{tu}} = \{g \in G_x \mid g^{p^n} \xrightarrow{n \rightarrow \infty} 1\}$. For the second equality, we have to show that for each $z \in \mathcal{B}(G)$, there exists y as in (a), such that $\mathcal{G}_x \cap \mathcal{G}_{z^+} \subset \mathcal{G}_{y^+}$. But every y , lying in the segment $[x, z] \subset \mathcal{B}(G)$, satisfies this property. \square

Proposition 1.8.16. *Let $\Phi : G \rightarrow \mathcal{G}$ be a quasi-logarithm defined over \mathcal{O} .*

(a) *For each $x \in \mathcal{B}(G)$, Φ induces measure preserving analytic isomorphisms $G_{x,\text{tu}} \xrightarrow{\sim} \mathcal{G}_{x,\text{tn}}$ and $G_{x^+} \xrightarrow{\sim} \mathcal{G}_{x^+}$ (with respect to measures $|\omega_G|$ and $|\omega_{\mathcal{G}}|$ chosen in 1.6.1).*

(b) *Φ induces a measure preserving analytic isomorphism $G(E)_{\text{tu}} \xrightarrow{\sim} \mathcal{G}(E)_{\text{tn}}$.*

(c) *Let ${}^0G \subset G$ be the biggest open subset $U \subset G$ such that $\Phi|_U$ is étale. Then ${}^0G(E)$ contains $G(E)_{\text{tu}}$.*

Proof. (a) For the proof, one can replace E by a finite unramified extension, so we can assume that G splits over E . By Lemma 1.8.7 (b), we have $\Phi(G_{x^+}) \subset \mathcal{G}_{x^+}$ for each $x \in \mathcal{B}(G)$, therefore by Lemma 1.8.15 (a), $\Phi(G_{x,\text{tu}}) \subset \mathcal{G}_{x,\text{tn}}$. Since $|\omega_G|(G_{x^+}) = |\omega_{\mathcal{G}}|(\mathcal{G}_{x^+})$, the second assertion follows from the first one.

Let us first show the assertion for a hyperspecial vertex $x \in \mathcal{B}(G)$. By Lemma 1.8.7 (b), Φ extends to the morphism $\Phi_x : \underline{G}_x \rightarrow \underline{\mathcal{G}}_x$ of schemes over \mathcal{O} , whose special fiber is a quasi-logarithm map $\overline{\Phi}_x : L_x \rightarrow \mathcal{L}_x$. By [BR, 9.1, 9.2, 9.3.3 and 6.3] (compare [BR, 9.4]), $\overline{\Phi}_x$ induces an isomorphism $\mathcal{U}(L_x) \xrightarrow{\sim} \mathcal{N}(\mathcal{L}_x)$. Moreover, there exists an open affine neighborhood $V \subset L_x$ of $\mathcal{U}(L_x)$ such that $\overline{\Phi}_x|_V$ is étale (see [BR, Thm 6.2 and 9.1]). Therefore by Hensel's lemma, Φ_x induces an analytic isomorphism $G_{x,\text{tu}} \xrightarrow{\sim} \mathcal{G}_{x,\text{tn}}$.

Since Φ_x is an algebraic morphism over \mathcal{O} , we get that $\Phi(gG_{x,r}) = \Phi(g) + \mathcal{G}_{x,r}$ for each $g \in G_{x,\text{tu}}$ and $r \in \mathbb{N}$. But $|\omega_G|(gG_{x,r}) = |\omega_{\mathcal{G}}|(\Phi(g) + \mathcal{G}_{x,r})$, and $\{gG_{x,r}\}_{g,r}$ form a basis of open neighborhoods of $G_{x,\text{tu}}$. Hence the analytic isomorphism $G_{x,\text{tu}} \xrightarrow{\sim} \mathcal{G}_{x,\text{tn}}$ is measure preserving.

It remains to show that for every $x, y \in \mathcal{B}(G)$, the assertions for x and y are equivalent. Moreover, we can assume that G_y is an Iwahori subgroup of G_x (compare the proof of Lemma 1.8.7). Then $G_{y,\text{tu}} = G_{x,\text{tu}} \cap G_y$ and $\mathcal{G}_{y,\text{tn}} = \mathcal{G}_{x,\text{tn}} \cap \mathcal{G}_y$ (see Lemma 1.8.15), so the assertion for x implies that for y . The opposite direction follows from equalities $G_{x,\text{tu}} = \cup_{g \in G_x} gG_{y,\text{tu}}g^{-1}$ and $\mathcal{G}_{x,\text{tn}} = \cup_{g \in G_x} \text{Ad } g(\mathcal{G}_{y,\text{tn}})$.

(b) By (a), we get that $\Phi(G(E)_{\text{tu}}) = \mathcal{G}(E)_{\text{tn}}$, and that the induced map $G(E)_{\text{tu}} \rightarrow \mathcal{G}(E)_{\text{tn}}$ is open. Thus we have to check that the restriction of Φ to $G(E)_{\text{tu}}$ is one-to-one.

Assume that $g_1, g_2 \in G(E)_{\text{tu}}$ satisfy $\Phi(g_1) = \Phi(g_2)$. Choose $x, y \in \mathcal{B}(G)$ such that $g_1 \in G_{x^+}$ and $g_2 \in G_{y^+}$ (use Lemma 1.8.15). By (a), Φ induces a measure preserving embedding $G_{x^+} \cap G_{y^+} \hookrightarrow \mathcal{G}_{x^+} \cap \mathcal{G}_{y^+}$. As measures of both sides are equal,

the last embedding is surjective. But $\Phi(g_1) = \Phi(g_2)$ belongs to $\mathcal{G}_{x+} \cap \mathcal{G}_{y+}$, hence there exists $g_3 \in G_{x+} \cap G_{y+}$ such that $\Phi(g_1) = \Phi(g_3) = \Phi(g_2)$. Since $\Phi|_{G_{x+}}$ and $\Phi|_{G_{y+}}$ are injective, we get that $g_1 = g_3 = g_2$, as claimed.

(c) We have to check that for each $g \in G(E)_{\text{tu}}$, the differential $d\Phi_g : T_g(G) \rightarrow T_{\Phi(g)}(\mathcal{G})$ is an isomorphism. But this follows from (b). \square

We finish this subsection with a result, which we will need later.

Lemma 1.8.17. (a) Let $\pi : \tilde{G} \rightarrow G$ be an isogeny (that is, a finite surjective quasi-isogeny) of order prime to p . Then $\pi(\tilde{G}(E)_{\text{tu}}) = G(E)_{\text{tu}}$.

(b) Let $\pi : \tilde{G} \rightarrow G$ be a surjective quasi-isogeny such that $S = \ker \pi$ is a torus split over E^{nr} . Then for every $x \in \mathcal{B}(\tilde{G})$, we have $\pi(\tilde{G}_x) = G_{\pi(x)}$.

Proof. (a) Since π is an isogeny, it identifies $\mathcal{B}(\tilde{G})$ with $\mathcal{B}(G)$. Thus we have to check that for each $x \in \mathcal{B}(\tilde{G}) = \mathcal{B}(G)$, we have $\pi(\tilde{G}_{x,\text{tu}}) = G_{x,\text{tu}}$. Since the order of π is prime to p , the corresponding map $\pi_x : \tilde{G}_x \rightarrow G_x$ of group schemes over \mathcal{O} is étale. Since the special fiber $\bar{\pi}_x : \bar{\tilde{G}}_x \rightarrow \bar{G}_x$ induces an isomorphism $\mathcal{U}(\bar{\tilde{G}}_x) \xrightarrow{\sim} \mathcal{U}(\bar{G}_x)$, the assertion follows from Hensel's lemma.

(b) Homomorphism π gives rise to an exact sequence $1 \rightarrow S_{\mathcal{O}} \rightarrow \tilde{G}_x \xrightarrow{\pi_x} G_{\pi(x)} \rightarrow 1$ of group schemes over \mathcal{O} . In particular, π_x is smooth. Passing to special fibers, we get an exact sequence $1 \rightarrow \bar{S} \rightarrow \bar{\tilde{G}}_x \xrightarrow{\bar{\pi}_x} \bar{G}_{\pi(x)} \rightarrow 1$ of groups over \mathbb{F}_q . Since \bar{S} is connected, we get that $H^1(\mathbb{F}_q, \bar{S}) = 1$, hence the map $\bar{\pi}_x(\mathbb{F}_q) = \pi_x(\mathbb{F}_q)$ is surjective. Therefore the surjectivity of $\pi_x(\mathcal{O})$ follows from Hensel's lemma. \square

Corollary 1.8.18. Let $\iota : G^{\text{sc}} \rightarrow G$ be a canonical map. Then for every $x \in \mathcal{B}(G)$ and an unramified maximal torus $T \subset G$, we have $G_x \subset \iota(G^{\text{sc}})(E) \cdot T(\mathcal{O})$.

Proof. Assume first that $G^{\text{der}} = G^{\text{sc}}$. Denote by q the projection $G \rightarrow G^{\text{ab}}$. Then we have to check that $q(G_x) \subset q(T(\mathcal{O}))$. Since $q(G_x) \subset G^{\text{ab}}(\mathcal{O})$, the assertion follows from part (b) of the lemma applied to the morphism $q|_T : T \rightarrow G^{\text{ab}}$.

For a general G , there exists a surjective quasi-isogeny $\pi : \tilde{G} \rightarrow G$ such that $\tilde{G}^{\text{der}} = \tilde{G}^{\text{sc}} (= G^{\text{sc}})$, and $\text{Ker } \pi$ is an induced torus splitting over E^{nr} (see [MS, Prop. 3.1]). Then for every $\tilde{x} \in \mathcal{B}(\tilde{G})$ such that $\pi(\tilde{x}) = x$ we have $\pi(\tilde{G}_{\tilde{x}}) = G_x$, so the assertion for G_x and T follows from that for $\tilde{G}_{\tilde{x}}$ and $\pi^{-1}(T) \subset \tilde{G}$. \square

2. ENDOSCOPIC DECOMPOSITION

2.1. Main Theorem.

In this subsection we will give two equivalent formulations of the main result of the paper.

2.1.1. Deligne–Lusztig representations. Let L be a connected reductive group over \mathbb{F}_q , $\bar{a} : \bar{T} \hookrightarrow L$ an embedding of a maximal torus of L , and $\bar{\theta} : \bar{T}(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$ a character. To this data Deligne and Lusztig [DL] associate a virtual representation $R_{\bar{a}(\bar{T})}^\theta$ of $L(\mathbb{F}_q)$. Moreover, if the torus $\bar{a}(\bar{T}) \subset L$ is elliptic and the character $\bar{\theta}$ is non-singular, then $\rho_{\bar{a}, \bar{\theta}} := (-1)^{\text{rk}_{\mathbb{F}_q}(L) - \text{rk}_{\mathbb{F}_q}(\bar{T})} R_{\bar{a}(\bar{T})}^\theta (= e(L) R_{\bar{a}(\bar{T})}^\theta)$ is a cuspidal representation (see [DL, Prop. 7.4 and Thm. 8.3]). In particular, $\rho_{\bar{a}, \bar{\theta}}$ is a genuine representation and not a virtual one. Moreover, $\rho_{\bar{a}, \bar{\theta}}$ is irreducible, if $\bar{\theta}$ is in general position.

2.1.2. Recall that there is an equivalence of categories $T \mapsto \bar{T}$ between tori over E splitting over E^{nr} and tori over \mathbb{F}_q . Moreover, every such T has a canonical \mathcal{O} -structure. We denote by $T(\mathcal{O})^+$ and $\mathcal{T}(\mathcal{O})^+$ the kernels of the reduction maps $\text{Ker}[T(\mathcal{O}) \rightarrow \bar{T}(\mathbb{F}_q)]$ and $\text{Ker}[\mathcal{T}(\mathcal{O}) \rightarrow \bar{\mathcal{T}}(\mathbb{F}_q)]$, respectively.

Notation 2.1.3. (a) Let G be a reductive group over E , T a torus over E splitting over E^{nr} , and $a : T \hookrightarrow G$ an embedding of a maximal elliptic torus of G .

For every vertex x of $\mathcal{B}(T)$, $a(x)$ is a vertex of $\mathcal{B}(G)$. Moreover, since T is elliptic, we have $\mathcal{B}(T) = \mathcal{B}(Z(G)^0)$. Thus the $Z(G)^0(E)$ -orbit of $a(x)$, hence also the parahoric subgroup $G_{a(x)}$ does not depend on x . Therefore we can denote $G_{a(x)}$ by G_a , and similarly for $\mathcal{G}_{a(x)}$, $G_{a(x)^+}$, $\mathcal{G}_{a(x)^+}$, $L_{a(x)}$ and $\mathcal{L}_{a(x)}$. We also set $\widetilde{G}_a := Z(G)(E)G_a$.

An embedding $a : T \hookrightarrow G$ induces an embedding $\bar{a} : \bar{T} \hookrightarrow L_a$ of a maximal elliptic torus of L_a .

(b) Let $\theta : T(E) \rightarrow \mathbb{C}^\times$ be a non-singular character (that is, θ is not orthogonal to any coroot of (G, T)), trivial on $T(\mathcal{O})^+$. Denote by $\bar{\theta} : \bar{T}(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$ the character of $\bar{T}(\mathbb{F}_q)$ defined by θ . Then there exists a unique representation $\rho_{a, \theta}$ of \widetilde{G}_a , whose central character is the restriction of θ , extending the inflation to G_a of the Deligne–Lusztig representation $\rho_{\bar{a}, \bar{\theta}}$ of $L_a(\mathbb{F}_q)$. We denote by $\pi_{a, \theta}$ the induced representation $\text{Ind}_{\widetilde{G}_a}^{G(E)} \rho_{a, \theta}$ of $G(E)$. Since for each irreducible factor $\rho' \subset \rho_{a, \theta}$, the induced representation $\text{Ind}_{\widetilde{G}_a}^{G(E)} \rho'$ is cuspidal and irreducible (see [MP2, Prop. 6.6]), we get that $\pi_{a, \theta}$ is a semisimple cuspidal representation of finite length, which is irreducible, if θ is in general position.

Definition 2.1.4. (a) Let $a : T \hookrightarrow G$ be an embedding of a maximal torus split over E^{nr} . We say that an element $\bar{t} \in \bar{T}(\mathbb{F}_q)$ is *a-strongly regular*, if \bar{t} is not fixed by a non-trivial element of the Weyl group $W(G, a(T)) \subset \text{Aut}(T_{E^{\text{nr}}}) = \text{Aut}(\bar{T}_{\mathbb{F}_q})$.

(b) Let $a^{\text{sc}} : T^{\text{sc}} \hookrightarrow G^{\text{sc}}$ be the lift of a . We say that G *satisfies property* $(vg)_a$, if G satisfies property (vg) (see Definition 1.8.10) and there exists an a^{sc} -strongly regular element $\bar{t} \in \bar{T}^{\text{sc}}(\mathbb{F}_q)$.

Notation 2.1.5. To each $\kappa \in \widehat{T}^\Gamma / Z(\widehat{G})^\Gamma$, an embedding $a_0 : T \hookrightarrow G$, and a character θ of $T(E)$ as in Notation 2.1.3, we associate an invariant generalized function

$$\chi_{a_0, \kappa, \theta} := e(G) \sum_a \langle \text{inv}(a_0, a), \kappa \rangle \chi(\pi_{a, \theta}) \in \mathcal{D}(G(E)).$$

Here a runs over a set of representatives of conjugacy classes of embeddings $T \hookrightarrow G$ which are stably conjugate to a_0 , and $\chi(\pi_{a, \theta})$ denotes the character of $\pi_{a, \theta}$.

Now we are ready to formulate our main result of the paper.

Theorem 2.1.6. *Let (a_0, κ, θ) be as in Notation 2.1.5. Assume that the characteristic of E is zero and G satisfies property $(vg)_{a_0}$. Then*

- (a) *The generalized function $\chi_{a_0, \kappa, \theta}$ is $\mathcal{E}_{([a_0], \kappa)}$ -stable.*
- (b) *For each inner twisting $\varphi : G \rightarrow G'$ and each embedding $a'_0 : T \hookrightarrow G'$, stably conjugate to a_0 , the generalized functions $\chi_{a_0, \kappa, \theta}$ on $G(E)$ and $\chi_{a'_0, \kappa, \theta}$ on $G'(E)$ are $(a_0, a'_0; \kappa)$ -equivalent.*

Remark 2.1.7. By Corollary 1.6.12 (b), assertion (a) is a particular case of (b). Moreover, if φ is not $\mathcal{E}_{([a_0], \kappa)}$ -admissible, then assertions (a) and (b) are equivalent (by Corollary 1.6.12 (c)).

Notation 2.1.8. (a) To each $a : T \hookrightarrow G$ and $\theta : T(E) \rightarrow \mathbb{C}^\times$ as in Notation 2.1.3 we associate a function $t_{a, \theta}$ on $G(E)$, supported on \widetilde{G}_a and equal to $\text{Tr } \rho_{a, \theta}$ there.

(b) Since $t_{a, \theta}$ is cuspidal, it follows from [HC1, Lem. 23] that for each $\gamma \in G^{\text{sr}}(E)$ and each compact open subgroup $K \subset G(E)$, the sum $\sum_{g \in D_b} t_{a, \theta}(g\gamma g^{-1})$, where $D_b := \widetilde{G}_a \backslash \widetilde{G}_a b K$, does not vanish only for finitely many $b \in \widetilde{G}_a \backslash G(E)/K$. Therefore the sum

$$F_{a, \theta}(\gamma) := \sum_{b \in \widetilde{G}_a \backslash G(E)/K} \left[\sum_{g \in D_b} t_{a, \theta}(g\gamma g^{-1}) \right]$$

stabilizes, and the resulting value is independent of K .

Explicitly, $F_{a, \theta}(\gamma) = \sum_{g \in \widetilde{G}_a \backslash \Omega} t_{a, \theta}(g\gamma g^{-1})$ for each sufficiently large compact modulo center subset $\Omega = \widetilde{G}_a \Omega K \subset G(E)$. In particular, $F_{a, \theta}$ is a locally constant invariant function on $G^{\text{sr}}(E)$.

(c) For each $\kappa \in \widehat{T}^\Gamma / Z(\widehat{G})^\Gamma$, put

$$F_{a_0, \kappa, \theta} := e(G) \sum_a \langle \text{inv}(a_0, a), \kappa \rangle F_{a, \theta},$$

where a runs over a set of representatives of conjugacy classes of embeddings $T \hookrightarrow G$ which are stably conjugate to a_0 .

Lemma 2.1.9. *Assume that the characteristic of E is zero. Then for each a and θ as in Notation 2.1.3, $F_{a,\theta}$ belongs to $L_{loc}^1(G(E))$, and the corresponding generalized function is equal to $\chi(\pi_{a,\theta})$.*

Proof. Since $\pi_{a,\theta}$ is cuspidal, the assertion is a combination of the theorem of Harish-Chandra ([HC1, Thm. 16]) and a formula for characters of induced representations. \square

For the next result, we will use Notation 1.7.3.

Theorem 2.1.10. *Under the assumptions of Theorem 2.1.6, let $\gamma \in G^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_\gamma^\Gamma / Z(\widehat{G})^\Gamma)$ be such that $F_{a_0,\kappa,\theta}(\gamma, \bar{\xi}) \neq 0$. Then*

- (i) $([a_\gamma], \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$;
- (ii) *if $\varphi : G \rightarrow G'$ is $(\mathcal{E}, [a_\gamma], \bar{\xi})$ -admissible, then for every stable conjugate $\gamma' \in G'(E)$ of γ we have $F'_{a'_0,\kappa,\theta}(\gamma', \bar{\xi}) = \left\langle \frac{\gamma, \gamma'; \bar{\xi}}{a, a'; \kappa} \right\rangle F_{a_0,\kappa,\theta}(\gamma, \bar{\xi})$.*

Lemma 2.1.11. *Theorem 2.1.10 is equivalent to Theorem 2.1.6.*

Proof. The equivalence follows from Lemma 2.1.9 and Proposition 1.7.4. More precisely, Proposition 1.7.4 (a) implies the equivalence between Theorem 2.1.6 (a) and Theorem 2.1.10 (i), while Proposition 1.7.4 (b) implies the equivalence between Theorem 2.1.6 (b) and a combination of Theorem 2.1.6 (a) and Theorem 2.1.10 (ii). \square

Remark 2.1.12. If the characteristic of E is positive, then it is not known that $\chi(\pi_{a,\theta})$ belongs to $L_{loc}^1(G(E))$. However the restriction $\chi(\pi_{a,\theta})|_{G^{\text{sr}}(E)}$ belongs to $L_{loc}^1(G^{\text{sr}}(E))$, therefore Proposition 1.7.4 implies that Theorem 2.1.10 for E is equivalent to an analog of Theorem 2.1.6 for restrictions $\chi_{a_0,\kappa,\theta}|_{G^{\text{sr}}(E)}$ and $\chi_{a'_0,\kappa,\theta}|_{G'^{\text{sr}}(E)}$.

Moreover, Theorem 2.1.10 for local fields of positive characteristic follows from that for local fields of characteristic zero by approximation arguments of [Ka3] and [De] (see [KV2]).

2.2. Stability of the restriction to $G(E)_{\text{tu}}$.

Starting from this subsection we will assume that the characteristic of E is zero. In this subsection we will strongly use definitions and results from Subsection 1.8.

2.2.1. Assumptions. Assume that G admits a quasi-logarithm map $\Phi : G \rightarrow \mathcal{G}$ defined over \mathcal{O} , \mathcal{G} admits an invariant pairing $\langle \cdot, \cdot \rangle$ non-degenerate over \mathcal{O} , and there exists $t \in \mathcal{T}(\mathcal{O})$, whose reduction $\bar{t} \in \overline{\mathcal{T}}(\mathbb{F}_q)$ is a_0 -strongly regular (see Definition 2.1.4 (a)).

Notation 2.2.2. (a) For every generalized function $F \in \mathcal{D}(G(E))$, denote by F_{tu} the restriction of $F|_{G(E)_{\text{tu}}}$ (see Notation 1.6.2 and Notation 1.8.14). Since $G(E)_{\text{tu}} \subset$

${}^0G(E)$ (see Proposition 1.8.16 (c)), we can consider F_{tu} as an element either of $\mathcal{D}(G(E))$ or of $\mathcal{D}({}^0G(E))$.

(b) Denote by ${}^0\Phi : {}^0G \rightarrow \mathcal{G}$ the restriction of Φ to 0G .

The goal of this subsection is to prove the following particular case of Theorem 2.1.6.

Theorem 2.2.3. *Let (a_0, κ, θ) be as in Notation 2.1.5. Under the assumptions of 2.2.1, the generalized functions $(\chi_{a_0, \kappa, \theta})_{\text{tu}}$ and $(\chi_{a'_0, \kappa, \theta})_{\text{tu}}$ are $(a_0, a'_0; \kappa)$ -equivalent. In particular, each $(\chi_{a_0, \kappa, \theta})_{\text{tu}}$ is $\mathcal{E}_{([a_0], \kappa)}$ -stable.*

Theorem 2.2.3 will be deduced in 2.2.13 from the corresponding statement about generalized functions on Lie algebras.

Notation 2.2.4. For every $a : T \hookrightarrow G$ as in Notation 2.1.3, we denote by $\overline{\Omega}_{a,t} \subset \mathcal{L}_a(\mathbb{F}_q)$ the $\text{Ad } L_a(\mathbb{F}_q)$ -orbit of $\overline{a}(\overline{t})$, by $\Omega_{a,t} \subset \mathcal{G}_a \subset \mathcal{G}(E)$ the preimage of $\overline{\Omega}_{a,t}$, and let $\delta_{a,t}$ and $\overline{\delta}_{a,t}$ be the characteristic functions of $\Omega_{a,t}$ and $\overline{\Omega}_{a,t}$, respectively.

Lemma 2.2.5. (a) *For each $y \in \Omega_{a,t}$, its stabilizer $G_y \subset G$ is G_a -conjugate to $a(T)$;*
 (b) *for each $y \in \Omega_{a,t}$ and $g \in G(E)$ such that $\text{Ad } g(y) \in \Omega_{a,t}$, we have $g \in \widetilde{G}_a$.*

Proof. (a) Since $\overline{t} \in \overline{\mathcal{T}}(\mathbb{F}_q)$ is a_0 -strongly regular, we see that $a(t) \in a(\mathcal{T}(E)) \subset \mathcal{G}(E)$ is strongly regular, hence $G_{a(t)} = a(T)$.

First we will show that for every $y \in a(t) + \mathcal{G}_{a+}$, we have $y \in \mathcal{G}^{\text{sr}}(E)$, and G_y is G_{a+} -conjugate to $a(T)$. By [DB, Lem. 2.2.2], it will suffice to prove that $y \in \mathcal{G}(E)$ is G -regular, and G_y splits over E^{nr} . For this we can replace E by an unramified extension, so we may assume that T splits over E . Under this assumption we will show that y is $G(E)$ -conjugate to an element of $a(t + \mathcal{T}(\mathcal{O})^+)$.

Choose an Iwahori subgroup $I \subset G_a$, containing $a(\mathcal{T}(\mathcal{O}))$, and let \mathcal{I}, I^+ and \mathcal{I}^+ be the corresponding Iwahori subalgebra, the pro-unipotent radical of I and the pro-nilpotent radical of \mathcal{I} , respectively. Since $\alpha(a(t)) \in \mathcal{O}^\times$ for each root α of $(G, a(T))$, it follows from direct calculations that every element of $a(t) + \mathcal{I}^+$ is I^+ -conjugate to an element of $a(t + \mathcal{T}(\mathcal{O})^+)$. But $y \in a(t) + \mathcal{G}_{a+} \subset a(t) + \mathcal{I}^+$, therefore we get the assertion in this case.

For an arbitrary $y \in \Omega_{a,t}$, there exists $h \in G_a$ such that $\text{Ad } h(y) \in a(t) + \mathcal{G}_{a+}$. So the general case follows from the previous one.

(b) Replacing y and $\text{Ad } g(y)$ by their G_a -conjugates, we can assume that $y \in a(t) + \mathcal{G}_{a+}$ and $\text{Ad } g(y) \in a(t) + \mathcal{G}_{a+}$. Then by the shown in (a), one can further replace y and $\text{Ad } g(y)$ by their G_{a+} -conjugates, so that both G_y and $G_{\text{Ad } g(y)} = gG_yg^{-1}$ equal $a(T)$. Thus $g \in \text{Norm}_G(a(T))$.

Since $\overline{\text{Ad } g(y)} = \overline{a}(\overline{t}) = \overline{y}$ is not fixed by a non-trivial element of the Weyl group $W(G, a(T))$, we get that $g \in a(T)(E)$. By Corollary 2.2.7 (a) below, g therefore belongs to \widetilde{G}_a , as claimed. \square

Lemma 2.2.6. *Let T be an unramified torus over E , and $S \subset T$ a maximal split subtorus. Then $T(E) = T(\mathcal{O})S(E)$.*

Proof. By a very particular case of Lemma 1.8.17 (b), the projection $T(\mathcal{O}) \rightarrow (T/S)(\mathcal{O})$ is surjective, therefore we have to check that $(T/S)(E) = (T/S)(\mathcal{O})$. Since T/S is anisotropic over E , the group $(T/S)(E)$ is compact. Hence $(T/S)(E)$ is contained in $(T/S)(E) \cap (T/S)(\mathcal{O}_{E^{\text{sep}}}) = (T/S)(\mathcal{O})$, as claimed. \square

Corollary 2.2.7. (a) $\widetilde{G}_a = a(T)(E)G_a$;

(b) G_a is the unique maximal compact subgroup of \widetilde{G}_a .

Proof. (a) Since $Z(G) \subset a(T)$, we get the inclusion $\widetilde{G}_a \subset a(T)(E)G_a$. It remains to show that $a(T)(E)$ is contained in \widetilde{G}_a . Let $S \subset T$ be the maximal split subtorus. Since $a(T) \subset G$ is elliptic, we get $a(S) \subset Z(G)$. Now the assertion follows from the inclusion $a(T(\mathcal{O})) \subset G_a$ and Lemma 2.2.6.

(b) Assume that $g \in \widetilde{G}_a$ belongs to a compact subgroup. Choose $g_a \in G_a$ and $z \in Z(G)(E) \subset a(T)(E)$ such that $g = g_a z$. Since G_a is compact and the sequence $\{g^n\}_n = \{g_a^n z^n\}_n \subset \widetilde{G}_a$ has a convergent subsequence, the sequence $\{z^n\}_n \subset a(T)(E)$ has a convergent subsequence. Hence z is contained in $a(T)(\mathcal{O}) \subset G_a$, thus $g \in G_a$. \square

Notation 2.2.8. It follows from Lemma 2.2.5 (b), that for each $x \in \mathcal{G}(E)$ there exists at most one coset $g \in \widetilde{G}_a \backslash G(E)$ such that $\delta_{a,t}(\text{Ad } g(x)) \neq 0$. Therefore

$$\Delta_{a,t}(x) := \sum_{g \in \widetilde{G}_a \backslash G(E)} \delta_{a,t}(\text{Ad } g(x))$$

is the characteristic function of an open and closed subset $\text{Ad } G(E)(\Omega_{t,a}) \subset \mathcal{G}(E)$. In particular, $\Delta_{a,t}$ lies in $L_{loc}^1(\mathcal{G}(E)) \subset \mathcal{D}(\mathcal{G}(E))$. Similarly to Notation 2.1.5, we define elements $\Delta_{a_0,\kappa,t} := e(G) \sum_a \langle \text{inv}(a_0, a), \kappa \rangle \Delta_{a,t} \in L_{loc}^1(\mathcal{G}(E))$ and $\Delta'_{a'_0,\kappa,t} := e(G') \sum_{a'} \langle \text{inv}(a'_0, a'), \kappa \rangle \Delta_{a',t} \in L_{loc}^1(\mathcal{G}'(E))$.

Lemma 2.2.9. $e(G)\Delta_{a_0,\kappa,t}$ is $(a_0, a'_0; \kappa)$ -equivalent to $e(G')\Delta'_{a'_0,\kappa,t}$.

Proof. By Proposition 1.7.4, we have to show that for each $x_0 \in \mathcal{G}^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G}_{x_0}^\Gamma / Z(\widehat{G})^\Gamma)$ such that $\Delta_{a_0,\kappa,t}(x_0, \bar{\xi}) \neq 0$, we have

- (i) $([a_{x_0}], \bar{\xi}) \in \text{Im } \Pi_{\mathcal{E}}$;
- (ii) if $\varphi : G \rightarrow G'$ is $(\mathcal{E}, [a_{x_0}], \bar{\xi})$ -admissible, then for every stable conjugate $x'_0 \in \mathcal{G}'(E)$ of x_0 , we have $e(G')\Delta'_{a'_0,\kappa,\theta}(x'_0, \bar{\xi}) = \left\langle \frac{x_0, x'_0; \bar{\xi}}{a_0, a'_0; \kappa} \right\rangle e(G)\Delta_{a_0,\kappa,\theta}(x_0, \bar{\xi})$.

Recall that

$$e(G)\Delta_{a_0,\kappa,t}(x_0, \bar{\xi}) = \sum_x \sum_a \langle \text{inv}(x_0, x), \bar{\xi} \rangle^{-1} \langle \text{inv}(a_0, a), \kappa \rangle \Delta_{a,t}(x),$$

where x runs over a set of representatives of $G(E) \setminus [x_0] \subset G(E) \setminus \mathcal{G}(E)$. We identify T with $a_0(T) \subset G$ and \mathcal{T} with $a_0(\mathcal{T}) \subset \mathcal{G}$. By Lemma 2.2.5 (a), the support of each $\Delta_{a,t}$ consists of elements, stably conjugate to $\mathcal{T}(E)$. Hence replacing x_0 by a stable conjugate, we can assume that $x_0 \in \mathcal{T}(E)$. Then $G_{x_0} = T$, $a_{x_0} = a_0$, and thus $\bar{\xi}$ is an element of $\widehat{T}^\Gamma / Z(\widehat{G})^\Gamma$.

For a stable conjugate x of x_0 , we have $\Delta_{a,t}(x) = 1$ if and only if $\text{inv}(x_0, x) = \text{inv}(a_0, a)$. Therefore $e(G)\Delta_{a_0,\kappa,t}(x_0, \bar{\xi}) = \sum_a \langle \text{inv}(a_0, a), \kappa \bar{\xi}^{-1} \rangle$. Since the latter sum is non-zero, we get that $\bar{\xi} = \kappa$, and $e(G)\Delta_{a_0,\kappa,t}(x_0, \bar{\xi}) = |\widehat{T}^\Gamma / Z(\widehat{G})^\Gamma|$. Since $\mathcal{E} = \mathcal{E}_{([a_0], \kappa)}$, we get that $([a_{x_0}], \bar{\xi}) = ([a_0], \kappa) \in \text{Im } \Pi_{\mathcal{E}}$, showing the assertion (i).

To show (ii), we can replace x'_0 by a stably conjugate $a'_0(a_0^{-1}(x_0)) \in a'_0(\mathcal{T}(E)) \subset \mathcal{G}'(E)$. Then the same arguments show that $e(G')\Delta'_{a'_0,\kappa,t}(x'_0, \bar{\xi}) = |\widehat{T}^\Gamma / Z(\widehat{G})^\Gamma|$. Now the required assertion $\langle \frac{x_0, x'_0; \bar{\xi}}{a_0, a'_0; \kappa} \rangle = 1$ follows from equalities $a_{x_0} = a_0$, $a_{x'_0} = a'_0$ and $\bar{\xi} = \kappa$. \square

2.2.10. Fourier transform. Fix an additive character $\psi : E \rightarrow \mathbb{C}^\times$ such that $\psi|_{\mathcal{O}}$ is non-trivial, but $\psi|_{\mathfrak{m}}$ is trivial.

(a) The pairing $\langle \cdot, \cdot \rangle$ and the measure $dx = |\omega_{\mathcal{G}}|$ on $\mathcal{G}(E)$ give rise to the Fourier transform $\mathcal{F} = \mathcal{F}(\psi, \langle \cdot, \cdot \rangle, dx)$ on $C_c^\infty(\mathcal{G}(E))$. Then \mathcal{F} induces Fourier transforms on $\mathcal{S}(\mathcal{G}(E))$ and $\mathcal{D}(\mathcal{G}(E))$ given by the formulas $\mathcal{F}(f dx) := \mathcal{F}(f) dx$ for each $f \in C_c^\infty(\mathcal{G}(E))$ and $\mathcal{F}(F)(\phi) := F(\mathcal{F}(\phi))$ for each $\phi \in \mathcal{S}(\mathcal{G}(E))$ and $F \in \mathcal{D}(\mathcal{G}(E))$.

(b) For each $f_1, f_2 \in C_c^\infty(\mathcal{G}(E))$ we have $\int_{\mathcal{G}(E)} f_1 \mathcal{F}(f_2) dx = \int_{\mathcal{G}(E)} \mathcal{F}(f_1) f_2 dx$. Therefore the embedding $C_c^\infty(\mathcal{G}(E)) \hookrightarrow \mathcal{D}(\mathcal{G}(E))$ commutes with the Fourier transform.

(c) For each parahoric subalgebra $\mathcal{G}_a \subset \mathcal{G}(E)$, we denote by $\overline{\mathcal{F}} = \overline{\mathcal{F}}(\overline{\psi}, \langle \cdot, \cdot \rangle_a, \mu)$ the Fourier transform on $\mathcal{L}_a(\mathbb{F}_q)$, where the character $\overline{\psi} : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ is induced by ψ , pairing $\langle \cdot, \cdot \rangle_a$ is induced by $\langle \cdot, \cdot \rangle$ (see Lemma 1.8.7 (a)) and $\mu(l) = 1$ for each $l \in L_a(\mathbb{F}_q)$.

Lemma 2.2.11. *Denote by \mathcal{I}^+ the pro-nilpotent radical of an Iwahori subalgebra of \mathcal{G} . Then for each $u \in G(E)_{\text{tu}}$, we have*

$$t_{a,\theta}(u) = \mathcal{F}(\delta_{a,t})(\Phi(u)) |\omega_{\mathcal{G}}| (\mathcal{I}^+)^{-1}.$$

Proof. First we claim that $\mathcal{F}(\delta_{a,t})(x) = 0$ for each $x \in \mathcal{G}(E) \setminus \mathcal{G}_a$, and $\mathcal{F}(\delta_{a,t})(x) = \overline{\mathcal{F}}(\overline{\delta}_{a,t})(\overline{x}) |\omega_{\mathcal{G}}| (\mathcal{G}_{a+})$ for each $x \in \mathcal{G}(E) \setminus \mathcal{G}_a$. Indeed, since $\delta_{a,t}$ vanishes outside of \mathcal{G}_a , we have an equality

$$\mathcal{F}(\delta_{a,t})(x) = \int_{\mathcal{G}(E)} \psi(\langle x, y \rangle) \delta_{a,t}(y) dy = \int_{\mathcal{G}_a} \psi(\langle x, y \rangle) \delta_{a,t}(y) dy$$

for each $x \in \mathcal{G}(E)$. Since $\delta_{a,t}(y + y') = \delta_{a,t}(y)$ for each $y' \in \mathcal{G}_{a+}$, we conclude that $\mathcal{F}(\delta_{a,t})(x)$ equals

$$\sum_z \psi(\langle x, z \rangle) \delta_{a,t}(z) \int_{\mathcal{G}_{a+}} \psi(\langle x, y \rangle) dy,$$

where z runs over a set of representatives of $\mathcal{G}_a/\mathcal{G}_{a+}$ in \mathcal{G}_a .

The assumptions on ψ and $\langle \cdot, \cdot \rangle$ imply that \mathcal{G}_a is the orthogonal complement of \mathcal{G}_{a+} with respect to the pairing $(x, y) \mapsto \psi(\langle x, y \rangle)$. Therefore $\mathcal{F}(\delta_{a,t})(x) = 0$ for each $x \notin \mathcal{G}_a$, and

$$\mathcal{F}(\delta_{a,t})(x) = \sum_{\bar{z} \in \mathcal{L}_a(\mathbb{F}_q)} \psi(\langle \bar{x}, \bar{z} \rangle) \bar{\delta}_{a,t}(\bar{z}) |\omega_{\mathcal{G}}|(\mathcal{G}_{a+}) = \bar{\mathcal{F}}(\bar{\delta}_{a,t})(\bar{x}) |\omega_{\mathcal{G}}|(\mathcal{G}_{a+})$$

for each $x \in \mathcal{G}_a$.

Now we are ready to prove the lemma. Assume first that $u \in G_{a,\text{tu}}$. It follows from Lemma 1.8.9 (b) that $\Phi(u) \in \mathcal{G}_a$ and that Φ induces a quasi-logarithm $\bar{\Phi}_a : L_a \rightarrow \mathcal{L}_a$ satisfying $\overline{\Phi(u)} = \bar{\Phi}_a(\bar{u})$. Therefore $\mathcal{F}(\delta_{a,t})(\Phi(u))$ equals $\bar{\mathcal{F}}(\bar{\delta}_{a,t})(\bar{\Phi}_a(\bar{u})) |\omega_{\mathcal{G}}|(\mathcal{G}_{a+})$. It now follows from a combination of Theorem A.1 (see Appendix A) and the equality $|\omega_{\mathcal{G}}|(\mathcal{G}_{a+}) = q^{-\frac{1}{2} \dim(L_a/\bar{T})} |\omega_{\mathcal{G}}|(\mathcal{I}^+)$ that $\mathcal{F}(\delta_{a,t})(\Phi(u))$ equals $|\omega_{\mathcal{G}}|(\mathcal{I}^+) \text{Tr } \rho_{\bar{a},\bar{\theta}}(\bar{u}) = |\omega_{\mathcal{G}}|(\mathcal{I}^+) t_{a,\theta}(u)$.

Finally, assume that $u \in G(E)_{\text{tu}} \setminus G_{a,\text{tu}}$. In this case, $t_{a,\theta}(u) = 0$. On the other hand, by Proposition 1.8.16, Φ induces bijections $G(E)_{\text{tu}} \xrightarrow{\sim} \mathcal{G}(E)_{\text{tn}}$ and $G_{a,\text{tu}} \xrightarrow{\sim} \mathcal{G}_{a,\text{tn}}$, therefore $\Phi(u) \in \mathcal{G}(E)_{\text{tn}} \setminus \mathcal{G}_{a,\text{tn}}$. Using the equality $\mathcal{G}_{a,\text{tn}} = \mathcal{G}(E)_{\text{tn}} \cap \mathcal{G}_a$ from Lemma 1.8.15 (b), we conclude that $\Phi(u) \notin \mathcal{G}_a$, hence $\mathcal{F}(\delta_{a,t})(\Phi(u)) = 0$. This completes the proof of the lemma. \square

Corollary 2.2.12. *For each a , we have $\chi(\pi_{a,\theta})_{\text{tu}} = {}^0\Phi^*(\mathcal{F}(\Delta_{a,t}))_{\text{tu}} |\omega_{\mathcal{G}}|(\mathcal{I}^+)^{-1}$.*

Proof. Note first that ${}^0\Phi$ is étale, hence smooth, therefore the pullback ${}^0\Phi^*(\mathcal{F}(\Delta_{a,t}))$ is defined. Consider generalized functions $t_{a,\theta} \in C_c^\infty(G(E)) \subset \mathcal{D}(G(E))$ and $\delta_{a,t} \in C_c^\infty(\mathcal{G}(E)) \subset \mathcal{D}(\mathcal{G}(E))$. In light of 2.2.10 (b), Lemma 2.2.11 implies the equality of generalized functions

$$(2.2.1) \quad (t_{a,\theta})_{\text{tu}} = {}^0\Phi^*(\mathcal{F}(\delta_{a,t}))_{\text{tu}} |\omega_{\mathcal{G}}|(\mathcal{I}^+)^{-1}.$$

Since $\pi_{a,\theta} = \text{Ind}_{\widetilde{G}_a}^{G(E)} \rho_{a,\theta}$ is admissible, it follows from the formula for characters of induced representations that

$$(2.2.2) \quad \chi(\pi_{a,\theta}) = \sum_{g \in \widetilde{G}_a \backslash G(E)} (\text{Int } g)^*(t_{a,\theta}).$$

Explicitly, $\chi(\pi_{a,\theta})(\phi) = \sum_{g \in \widetilde{G}_a \backslash G(E)} (\text{Int } g)^*(t_{a,\theta})(\phi)$ for each $\phi \in \mathcal{S}(G(E))$, where only finitely many terms in the sum are non-zero. On the other hand, by the very

definition of $\Delta_{a,t}$, we have

$$(2.2.3) \quad \Delta_{a,t} = \sum_{g \in \widetilde{G}_a \backslash G(E)} (\text{Ad } g)^*(\delta_{a,t}).$$

Since $\langle \cdot, \cdot \rangle$ and Φ are G -equivariant, we get the equality

$$(2.2.4) \quad {}^0\Phi^*(\mathcal{F}((\text{Ad } g)^*\delta_{a,t})) = {}^0\Phi^*(\text{Ad } g)^*(\mathcal{F}(\delta_{a,t})) = (\text{Int } g)^*{}^0\Phi^*(\mathcal{F}(\delta_{a,t})).$$

Now our corollary is an immediate consequence of equalities (2.2.1)–(2.2.4). \square

2.2.13. Proof of Theorem 2.2.3. Let $\Phi' : G' \rightarrow \mathcal{G}'$ and $\langle \cdot, \cdot \rangle'$ be the quasi-logarithm map and the pairing on \mathcal{G}' induced by Φ and $\langle \cdot, \cdot \rangle$, respectively (see Lemma 1.8.9 (a)). We denote by $\mathcal{F} = \mathcal{F}(\psi, \langle \cdot, \cdot \rangle', |\omega_{\mathcal{G}'}|)$ the corresponding Fourier transform $\mathcal{G}'(E)$ and by \mathcal{I}^+ the pro-nilpotent radical of an Iwahori subalgebra of $\mathcal{G}'(E)$.

By Lemma 2.2.9, $e(G)\Delta_{a_0,\kappa,t}$ is $(a_0, a'_0; \kappa)$ -equivalent to $e(G')\Delta'_{a'_0,\kappa,t}$. Using the equality $e(G)e(G') = e'(G)e'(G')$, it follows from Theorem B.1.2 (see Appendix B) that $\mathcal{F}(\Delta_{a_0,\kappa,t})$ is $(a_0, a'_0; \kappa)$ -equivalent to $\mathcal{F}(\Delta'_{a'_0,\kappa,t})$. Hence by Corollary 1.6.16, the pullback ${}^0\Phi^*(\mathcal{F}(\Delta_{a_0,\kappa,t}))$ is $(a_0, a'_0; \kappa)$ -equivalent to ${}^0\Phi'^*(\mathcal{F}(\Delta'_{a'_0,\kappa,t}))$. By Corollary 2.2.12, we thus get that $|\omega_{\mathcal{G}}|(\mathcal{I}^+)(\chi_{a_0,\kappa,\theta})_{\text{tu}}$ and $|\omega_{\mathcal{G}'}|(\mathcal{I}'^+)(\chi_{a'_0,\kappa,\theta})_{\text{tu}}$ are $(a_0, a'_0; \kappa)$ -equivalent. Since $|\omega_{\mathcal{G}}|(\mathcal{I}^+) = |\omega_{\mathcal{G}'}|(\mathcal{I}'^+)$ (see, for example, [Ko4, p. 632]), the assertion follows. \square

2.3. Reduction formula.

In this subsection we will assume that $G^{\text{der}} = G^{\text{sc}}$. Our goal is to rewrite character $\chi(\pi_{a,\theta})$ in terms of restrictions to topologically unipotent elements of the corresponding characters of the centralizers $G_\delta(E)$.

Lemma 2.3.1. *Assume that $G^{\text{der}} = G^{\text{sc}}$. Then*

- (a) *For each semisimple element $\delta \in G$, the centralizer G_δ is connected.*
- (b) *The stabilizer in $G(E)$ of each $x \in \mathcal{B}(G)$ is G_x .*

Proof. (a) was shown in [St, Cor. 8.5] when G is semisimple, and in [Ko3, pp. 788–789] in the general case.

(b) When G is semisimple, the result was proved in [BT, Prop. 4.6.32]. For a general G , we can replace E by an unramified extension so that G is split over E . Choose a split maximal torus $T \subset G$ such that $x \in \mathcal{B}(T)$. Since $\text{Stab}_{G(E)}(x)$ is compact, we see as in Corollary 1.8.18 that $\text{Stab}_{G(E)}(x)$ is contained in $G^{\text{der}}(E)T(\mathcal{O})$. Since $T(\mathcal{O}) \subset \text{Stab}_{G(E)}(x)$, we get that $\text{Stab}_{G(E)}(x)$ is contained in $\text{Stab}_{G^{\text{der}}(E)}(x)T(\mathcal{O})$, hence (use [BT, Prop. 4.6.32]) in $(G^{\text{der}})_x T(\mathcal{O}) = G_x$, as claimed. \square

Notation 2.3.2. (a) We will call an element $\gamma \in G(E)$ *compact*, if it generates a relatively compact subgroup of $G(E)$.

(b) We will call an element $\gamma \in G(E)$ *topologically unipotent*, if the sequence $\{\gamma^{p^n}\}_n$ converges to 1.

Corollary 2.3.3. (a) *The set of compact elements of $G(E)$ is $\cup_{x \in \mathcal{B}(G)} G_x$.*

(b) *The set of topologically unipotent elements of $G(E)$ is $G(E)_{\text{tu}}$ (see Notation 1.8.14).*

Proof. As each G_x is compact and every compact element of $G(E)$ stabilizes a point of $\mathcal{B}(G)$, (a) follows from Lemma 2.3.1 (b). Since every topologically unipotent element is compact, (b) follows from (a) and the fact that $G_{x,\text{tu}}$ is set of all topologically unipotent elements of G_x . \square

The following result is a straightforward generalization of [Ka2, Lem 2, p. 226].

Lemma 2.3.4. *For every compact element $\gamma \in G(E)$, there exists a unique decomposition $\gamma = \delta u = u\delta$ such that δ is of finite order prime to p , and u is topologically unipotent. In particular, this decomposition is compatible with conjugation and field extensions.*

Notation 2.3.5. The decomposition $\gamma = \delta u$ from Lemma 2.3.4 is called the *topological Jordan decomposition* of γ .

Remark 2.3.6. If $\delta \in G(E)$ is an element of finite order prime to p , then δ is automatically semisimple.

The goal of this subsection is to prove the following result.

Proposition 2.3.7. *For every embedding $a : T \hookrightarrow G$ and a compact element $\gamma \in G(E)$ with topological Jordan decomposition $\gamma = \delta u$, we have the following formula*

$$(2.3.1) \quad e(G)F_{a,\theta}(\gamma) = e(G_\delta) \sum_b \theta(b^{-1}(\delta)) F_{b,\theta}(u).$$

Here b runs over a set of representatives of conjugacy classes of embeddings $T \hookrightarrow G_\delta$, whose composition with the inclusion $G_\delta \hookrightarrow G$ is conjugate to a .

First we need to prove two preliminary results.

Lemma 2.3.8. (a) *For each vertex x of $\mathcal{B}(G)$, we have $e(L_x) = e(G)$.*

(b) *Let $\delta \in G(E)$ be an element of finite order, prime to p . Then the centralizer G_δ splits over E^{nr} , and the building $\mathcal{B}(G_\delta)$ is canonically identified with the set of invariants $\mathcal{B}(G)^\delta \subset \mathcal{B}(G)$.*

(c) *For each $x \in \mathcal{B}(G_\delta) \subset \mathcal{B}(G)$, the parahoric subgroup $(G_\delta)_x$ is a subgroup of finite index in $(G_x)_\delta$, and the canonical map $(G_x)_\delta \hookrightarrow G_x \rightarrow L_x(\mathbb{F}_q)$ induces isomorphisms $(G_x)_\delta / (G_\delta)_{x^+} \xrightarrow{\sim} (L_x)_{\bar{\delta}}(\mathbb{F}_q)$ and $(G_\delta)_x / (G_\delta)_{x^+} \xrightarrow{\sim} (L_x)_{\bar{\delta}}^0(\mathbb{F}_q)$.*

(d) *Let $\delta, \delta' \in G_x$ be two elements of finite orders prime to p . Then δ and δ' are G_x -conjugate if and only if their reductions $\bar{\delta}, \bar{\delta}' \in L_x(\mathbb{F}_q)$ are $L_x(\mathbb{F}_q)$ -conjugate.*

Proof. (a) For each $x \in \mathcal{B}(G)$, the maximal split torus of L_x is the reduction of that of G , therefore $\mathrm{rk}_{\mathbb{F}_q}(L_x) = \mathrm{rk}_E(G)$. If, moreover, x is a vertex, then $\mathrm{rk}_{\mathbb{F}_q}(Z(L_x)^0) = \mathrm{rk}_E(Z(G)^0)$, hence $e(L_x) = e(G)$.

(b) The second assertion is shown in [PY]. For the first, recall that $G_{E^{\mathrm{nr}}}$ splits, hence there exists a split maximal torus $T \subset G_{E^{\mathrm{nr}}}$. Choose $g \in G(\overline{E})$ such that $g\delta g^{-1} \in T(\overline{E})$. Then $g\delta g^{-1}$ is of finite order, prime to p , therefore it follows from Hensel's lemma that $g\delta g^{-1} \in T(\mathcal{O}_{E^{\mathrm{nr}}}) \subset T(E^{\mathrm{nr}})$. Hence g gives rise to a cocycle $\sigma \mapsto g^{-1\sigma}g \in G_\delta(\overline{E})$ over E^{nr} . Since $H^1(E^{\mathrm{nr}}, G_\delta) = 0$, there exists $h \in G_\delta(\overline{E})$ such that $h^{-1\sigma}h = g^{-1\sigma}g$ for each $\sigma \in \mathrm{Gal}(\overline{E}/E^{\mathrm{nr}})$. It follows that $gh^{-1} \in G(E^{\mathrm{nr}})$, and $(gh^{-1})\delta(gh^{-1})^{-1} = g\delta g^{-1} \in T(E^{\mathrm{nr}})$. Therefore $(gh^{-1})^{-1}T(gh^{-1})$ is a split maximal torus of $(G_\delta)_{E^{\mathrm{nr}}}$.

(c) As $(G_x)_\delta \subset G_\delta(E)$ is the stabilizer of $x \in \mathcal{B}(G_\delta)$, it is compact. Therefore $(G_\delta)_x$ is a subgroup of finite index in $(G_x)_\delta$, thus the corresponding group scheme \underline{G}_{δ_x} over \mathcal{O} is the connected component of $(\underline{G}_x)_\delta$. In particular, $(\underline{G}_x)_\delta$ is smooth over \mathcal{O} . Therefore by Hensel's lemma, the reduction map $G_x \rightarrow \overline{G}_x(\mathbb{F}_q)$ surjects $(G_x)_\delta = (\underline{G}_x)_\delta(\mathcal{O})$ onto $(\overline{G}_x)_\delta(\mathbb{F}_q) = (\underline{G}_x)_\delta(\mathbb{F}_q)$.

As L_x is the quotient $\overline{G}_x/R_u(\overline{G}_x)$, we see that $(L_x)_{\overline{\delta}}$ is the quotient $(\overline{G}_x)_\delta/R_u(\overline{G}_x)_\delta$. Since $R_u(\overline{G}_x)_\delta$ is a connected group over \mathbb{F}_q , Lang's theorem implies that the projection $\overline{G}_x(\mathbb{F}_q) \rightarrow L_x(\mathbb{F}_q)$ surjects $(\overline{G}_x)_{\overline{\delta}}(\mathbb{F}_q)$ onto $(L_x)_{\overline{\delta}}(\mathbb{F}_q)$. Therefore the projection $G_x \rightarrow L_x(\mathbb{F}_q)$ induces a surjection $(G_x)_\delta \rightarrow (L_x)_{\overline{\delta}}(\mathbb{F}_q)$, whose kernel is $(G_x)_\delta \cap G_{x^+} = (G_\delta)_{x^+}$. This shows the first isomorphism, while the proof of the second one is similar but easier.

(d) The “only if” assertion is clear. Assume now that $\overline{\delta}$ and $\overline{\delta}'$ are $L_x(\mathbb{F}_q)$ -conjugate. Let us first show that δ and δ' are $\underline{G}_x(\mathcal{O}_{E^{\mathrm{nr}}})$ -conjugate. For this we can replace E by an unramified extension, so that G , G_δ and $G_{\delta'}$ are split over E (use (b)). Since δ lies in G_x , we get that x belongs to $\mathcal{B}(G)^\delta = \mathcal{B}(G_\delta)$ (use (b)). Therefore there exists a split maximal torus $T \subset G_\delta \subset G$ such that $x \in \mathcal{B}(T)$. Similarly, there exists a split maximal torus $T' \subset G_{\delta'} \subset G$ such that $x \in \mathcal{B}(T')$. By a property of buildings, there exists $g \in G_x$ such that $gTg^{-1} = T'$. Replacing δ' by $g^{-1}\delta'g$, we may assume that $\delta, \delta' \in T(E)$.

Next we observe that the projection $\mathrm{Norm}_{G_x}(T) \rightarrow \mathrm{Norm}_{L_x(\mathbb{F}_q)}(\overline{T})$ is surjective. Indeed, for each $\overline{g} \in \mathrm{Norm}_{L_x(\mathbb{F}_q)}(\overline{T}) \subset L_x(\mathbb{F}_q)$, choose a representative $g \in G_x$. Then $\overline{gTg^{-1}} = \overline{T}$, hence by [DB, Lem 2.2.2], there exists $h \in G_{x^+}$ such that $h(gTg^{-1})h^{-1} = T$. In other words, $hg \in \mathrm{Norm}_{G_x}(T)$ is a preimage of \overline{g} .

By the assumption, $\overline{\delta}, \overline{\delta}' \in \overline{T}(\mathbb{F}_q)$ are conjugate in $L_x(\mathbb{F}_q)$, therefore they are conjugate in $\mathrm{Norm}_{L_x(\mathbb{F}_q)}(\overline{T})$. Hence there exists $g \in \mathrm{Norm}_{G_x}(T)$ such that $\overline{g^{-1}\delta'g} = \overline{\delta}$. But the projection $T(\mathcal{O}) \rightarrow \overline{T}(\mathbb{F}_q)$ defines a bijection between elements of $T(\mathcal{O})$

of finite order prime to p and elements of $\overline{T}(\mathbb{F}_q)$. Hence $g^{-1}\delta'g = \delta$, implying that δ and δ' are conjugate by an element of $\underline{G}_x(\mathcal{O}_{E^{\text{nr}}})$.

To show that δ and δ' are conjugate by G_x , consider the closed subscheme Z (resp. Z') of \underline{G}_x (resp. L_x) consisting of elements g (resp. \bar{g}) such that $g\delta g^{-1} = \delta'$ (resp. $\bar{g}\bar{\delta}\bar{g}^{-1} = \bar{\delta}'$). By the assumption, $Z'(\mathbb{F}_q) \neq \emptyset$, and we have to show that $Z(\mathcal{O}) \neq \emptyset$. By the shown above, $Z(\mathcal{O}_{E^{\text{nr}}}) \neq \emptyset$. Thus Z and $(\underline{G}_x)_\delta$ are isomorphic over $\mathcal{O}_{E^{\text{nr}}}$. In particular, Z is smooth over \mathcal{O} , thus by Hensel's lemma it suffice to show that the projection $Z(\mathbb{F}_q) \rightarrow Z'(\mathbb{F}_q)$ is surjective.

Denote by $\overline{Z} \subset \overline{G}_x$ the special fiber of Z . Since all fibers of the projection $\overline{Z} \rightarrow Z'$ are principal homogeneous spaces for the connected group $R_u(\overline{G}_x)_\delta$, the surjectivity of the projection $Z(\mathbb{F}_q) = \overline{Z}(\mathbb{F}_q) \rightarrow Z'(\mathbb{F}_q)$ follows from Lang's theorem. \square

Lemma 2.3.9. *For every $\gamma \in G_a$ with topological Jordan decomposition $\gamma = \delta u$, we have an equality*

$$(2.3.2) \quad e(G)t_{a,\theta}(\gamma) = e(G_\delta) \sum_b \sum_{h \in (G_\delta)_b \backslash (G_a)_\delta} \theta(b^{-1}(\delta)) t_{b,\theta}(huh^{-1}),$$

where the b runs over a set of representatives of conjugacy classes of embeddings $T \hookrightarrow G_\delta$, which are G_a -conjugate to $a : T \hookrightarrow G$.

Proof. We start from the following claim

Claim 2.3.10. *The correspondence $b \mapsto \bar{b}$ induces a bijection between the set of conjugacy classes of embeddings $T \hookrightarrow G_\delta$ which are G_a -conjugate to $a : T \hookrightarrow G$ and the set of conjugacy classes of embeddings $\bar{b} : \bar{T} \hookrightarrow (L_a)_{\bar{\delta}}$, which are $L_a(\mathbb{F}_q)$ -conjugate to $\bar{a} : \bar{T} \hookrightarrow L_a$.*

Proof. For simplicity of notation, we identify T with $a(T)$ and \bar{T} with $\bar{a}(\bar{T})$. Then the maps $b \mapsto t := b^{-1}(\delta)$ and $\bar{b} \mapsto \bar{t} := \bar{b}^{-1}(\bar{\delta})$ identify our sets with the sets of elements $t \in T(E)$, which are G_a -conjugate to δ , and elements $\bar{t} \in \bar{T}(\mathbb{F}_q)$, which are $L_a(\mathbb{F}_q)$ -conjugate to $\bar{\delta}$, respectively. Since the reduction map $t \mapsto \bar{t}$ induces a bijection between elements of $T(E)$ of finite order prime to p and elements of $\bar{T}(\mathbb{F}_q)$, we get the injectivity. The surjectivity follows from Lemma 2.3.8 (d). \square

Now we are ready to prove the lemma. By Lemma 2.3.8 and Claim 2.3.10, the right hand side of (2.3.2) equals

$$(2.3.3) \quad e((L_a)_{\bar{\delta}}^0) \sum_{\bar{b}} \sum_{\bar{h} \in (L_a)_{\bar{\delta}}^0(\mathbb{F}_q) \backslash (L_a)_{\bar{\delta}}(\mathbb{F}_q)} \bar{\theta}(\bar{b}^{-1}(\bar{\delta})) \text{Tr } \rho_{\bar{b},\bar{\theta}}(\bar{h}\bar{u}\bar{h}^{-1}),$$

where \bar{b} runs over a set of representatives of $(L_a)_{\bar{\delta}}(\mathbb{F}_q)$ -conjugacy classes of embeddings $\bar{T} \hookrightarrow (L_a)_{\bar{\delta}}$, which are $L_a(\mathbb{F}_q)$ -conjugate to \bar{a} .

Next note that (2.3.3) can be rewritten as

$$(2.3.4) \quad e((L_a)_\delta^0) \sum_{\bar{b}: T \hookrightarrow (L_a)_\delta^0} \theta(\bar{b}^{-1}(\bar{\delta})) \operatorname{Tr} \rho_{\bar{b}, \bar{\theta}}(\bar{u}),$$

where \bar{b} runs over a set of representatives of conjugacy classes of embeddings, $L_a(\mathbb{F}_q)$ -conjugate to \bar{a} . By the formula of Deligne–Lusztig [DL, Thm 4.2], (2.3.4) equals $e(L_a) \operatorname{Tr} \rho_{\bar{a}, \bar{\theta}}(\bar{\gamma})$. Hence it is equal to $e(G)t_{a, \theta}(\gamma)$, as claimed. \square

2.3.11. Proof of Proposition 2.3.7. Notice first that the map $b \mapsto b^{-1}(\delta)$ embeds the set of conjugacy classes of embeddings $b : T \hookrightarrow G_\delta$, conjugate to $a : T \hookrightarrow G$, into the finite set $\{t \in T(E) \mid t^{\operatorname{ord} \delta} = 1\}$. Therefore the sum in (2.3.1) is finite.

Fix a set of representatives $J \subset G(E)$ of double classes $\widetilde{G}_a \backslash G(E) / G_\delta(E)$. For every $h \in J$, put $\gamma_h = h\gamma h^{-1}$, and let $\gamma_h = \delta_h u_h$ be the topological Jordan decomposition of γ_h .

By [HC1, Lem. 23] (compare Notation 2.1.8 (b)), for each sufficiently large compact modulo center \widetilde{G}_a -bi-invariant subset $\Omega \subset G(E)$, we have

$$F_{a, \theta}(\gamma) = e(G) \sum_{g \in \widetilde{G}_a \backslash \Omega} t_{a, \theta}(g\gamma g^{-1}).$$

Since $G(E)$ decomposes as a disjoint union $\sqcup_{h \in J} \widetilde{G}_a h G_\delta(E) = \sqcup_{h \in J} \widetilde{G}_a G_{\delta_h}(E) h$, we have a finite decomposition $\widetilde{G}_a \backslash \Omega = \sqcup_{h \in J} \widetilde{G}_a \backslash [\widetilde{G}_a G_{\delta_h}(E) h \cap \Omega]$. Therefore

$$F_{a, \theta}(\gamma) = e(G) \sum_{h \in J} \sum_{g \in \widetilde{G}_a \backslash [\widetilde{G}_a G_{\delta_h}(E) h \cap \Omega]} t_{a, \theta}(g\gamma g^{-1}).$$

Using the identifications

$$\widetilde{G}_a \backslash [\widetilde{G}_a G_{\delta_h}(E) h \cap \Omega] = (\widetilde{G}_a)_{\delta_h} \backslash [G_{\delta_h}(E) h \cap \Omega] = (\widetilde{G}_a)_{\delta_h} \backslash [G_{\delta_h}(E) \cap \Omega h^{-1}] h,$$

we get that $F_{a, \theta}(\gamma)$ equals

$$(2.3.5) \quad \sum_{h \in J} e(G) \sum_{g \in (\widetilde{G}_a)_{\delta_h} \backslash [G_{\delta_h}(E) \cap \Omega h^{-1}]} t_{a, \theta}(g\gamma_h g^{-1}).$$

Using Corollary 2.2.7, we see that for each embedding $b : T \hookrightarrow G_{\delta_h}$ which is G_a -conjugate to a , the group $(\widetilde{G}_{\delta_h})_b$ is contained in $(\widetilde{G}_a)_{\delta_h}$ and we have a natural isomorphism $(G_{\delta_h})_b \backslash (G_a)_{\delta_h} \cong (\widetilde{G}_{\delta_h})_b \backslash (\widetilde{G}_a)_{\delta_h}$. Then by Lemma 2.3.9 applied to $\gamma_h = \delta_h u_h$, the contribution of each $h \in J$ to (2.3.5) equals

$$(2.3.6) \quad e(G_{\delta_h}) \sum_{b: T \hookrightarrow G_{\delta_h}} \theta(b^{-1}(\delta_h)) \sum_{g \in (\widetilde{G}_{\delta_h})_b \backslash [G_{\delta_h}(E) \cap \Omega h^{-1}]} t_{b, \theta}(g u_h g^{-1}),$$

where b runs over the a set of representatives of conjugacy classes of embeddings, which are G_a -conjugate to $a : T \hookrightarrow G$.

Conjugating by h^{-1} , we can rewrite (2.3.6) in the form

$$(2.3.7) \quad e(G_\delta) \sum_{b: T \hookrightarrow G_\delta} \theta(b^{-1}(\delta)) \sum_{g \in \widetilde{(G_\delta)_b} \setminus [G_\delta(E) \cap h^{-1}\Omega]} t_{b,\theta}(gug^{-1}),$$

where b runs over a set of representatives of conjugacy classes of embeddings such that $a = gbg^{-1}$ for some $g \in \widetilde{G_a}hG_\delta(E)$. In particular, b has a non-trivial contribution to (2.3.7) only for a unique $h \in J$, which we denote by h_b . It follows that $F_{a,\theta}(\gamma)$ equals

$$(2.3.8) \quad e(G_\delta) \sum_{b: T \hookrightarrow G_\delta} \theta(b^{-1}(\delta)) \sum_{g \in \widetilde{(G_\delta)_b} \setminus [G_\delta(E) \cap h_b^{-1}\Omega]} t_{b,\theta}(gug^{-1}),$$

where b runs over a (finite) set of representatives of conjugacy classes to embeddings, which are $G(E)$ -conjugate to a .

Replacing b 's by their $G_\delta(E)$ -conjugates, we can assume that $a = gbg^{-1}$ for some $g \in \widetilde{G_a}h_b$. Then $h_b(G_\delta)_bh_b^{-1} \subset G_a$, hence $h_b(\widetilde{(G_\delta)_b})h_b^{-1} \subset \widetilde{G_a}$ (use Corollary 2.2.7). It follows that the subset $G_\delta(E) \cap h_b^{-1}\Omega \subset G_\delta(E)$ is compact modulo center, $(\widetilde{(G_\delta)_b})_b$ -invariant from the left, and $h_b(\widetilde{(G_\delta)_b})h_b^{-1}$ -invariant from the right. Since the number of b 's is finite, it follows from [HC1, Lem. 23], that for each sufficiently large $\Omega \subset G(E)$, the contribution of each b to (2.3.8) equals $\theta(b^{-1}(\delta))F_{b,\theta}(u)$. This completes the proof. \square

2.4. Endoscopy for G and G_δ .

Let G be a connected reductive group over E such that $G^{\text{der}} = G^{\text{sc}}$, $\delta \in G(E)$ a semisimple element, and $\iota : G_\delta \hookrightarrow G$ the canonical embedding. In this subsection we will compare endoscopic triples for G and for G_δ .

2.4.1. (a) Similarly to 1.2.4 (b), there exists a natural embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{G_\delta})$. Indeed, every maximal torus T of G_δ is a maximal torus of G , and the set of roots of (G_δ, T) equals the set of those roots of (G, T) which vanish on δ . Hence the set of coroots (hence also of roots) of $(\widehat{G_\delta}, \widehat{T})$ is naturally a subset of those of $(\widehat{G}, \widehat{T})$. Thus $Z(\widehat{G})$ is naturally a subgroup of $Z(\widehat{G_\delta})$.

(b) Fix an embedding $a_\delta : T \hookrightarrow G_\delta$ of a maximal torus and $\kappa \in \widehat{T}^\Gamma / Z(\widehat{G})^\Gamma$. Set $a := \iota \circ a_\delta : T \hookrightarrow G$, let $\bar{\kappa} \in \pi_0(\widehat{T}^\Gamma / Z(\widehat{G})^\Gamma)$ and $\bar{\kappa}' \in \pi_0(\widehat{T}^\Gamma / Z(\widehat{G_\delta})^\Gamma)$ be the classes of κ , and put $\mathcal{E} := \mathcal{E}_{([a], \kappa)} = (H, [\eta], \bar{s})$ and $\mathcal{E}' := \mathcal{E}_{([a_\delta], \kappa)} = (H', [\eta'], \bar{s}')$.

Also we fix embeddings $c' : T \hookrightarrow H'$ and $c : T \hookrightarrow H$ such that $\Pi_{\mathcal{E}}([c]) = ([a], \bar{\kappa})$ and $\Pi_{\mathcal{E}'}([c']) = ([a_\delta], \bar{\kappa}')$ (see 1.3.8 (b)).

Lemma 2.4.2. (a) *There is a natural Γ -equivariant embedding $Z(\widehat{H}) \hookrightarrow Z(\widehat{H}')$ mapping $Z(\widehat{G})$ into $Z(\widehat{G}_\delta)$ and an embedding $W(H') \hookrightarrow W(H)$, both of which depend on c and c' . The induced map $\pi_0(Z(\widehat{H})^\Gamma/Z(\widehat{G})^\Gamma) \rightarrow \pi_0(Z(\widehat{H}')^\Gamma/Z(\widehat{G}_\delta)^\Gamma)$ sends \bar{s} to \bar{s}' .*

(b) *There is a natural map $[b'] \mapsto [b]$, depending on c and c' , from the set of stable conjugacy classes of embeddings of maximal tori $S \hookrightarrow H'$ to those of embeddings $S \hookrightarrow H$.*

(c) *In the notation of (b), we have $[b]_G = \iota \circ [b']_{G_\delta}$, and $Z_{[b]} : Z(\widehat{H}) \hookrightarrow \widehat{S}$ is the restriction of $Z_{[b']} : Z(\widehat{H}') \hookrightarrow \widehat{S}$ (see (a)). In particular, $\bar{\kappa}_{[b']} \in \pi_0(\widehat{S}^\Gamma/Z(\widehat{G}_\delta)^\Gamma)$ is the image of $\bar{\kappa}_{[b]} \in \pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)$.*

(d) *Let $[b'_i]$ be two stable conjugacy classes of embeddings of maximal tori $T_i \hookrightarrow H'$, and let $[b_i]$ be the corresponding stable conjugacy classes of embeddings $T_i \hookrightarrow H$. Then the following diagram is commutative*

$$\begin{array}{ccc} Z(\widehat{H}) & \xrightarrow{\iota([b_1], [b_2])} & \{(T_1 \times T_2)/Z(G)\}^\wedge \\ \downarrow & & \uparrow \\ Z(\widehat{H}') & \xrightarrow{\iota([b'_1], [b'_2])} & \{(T_1 \times T_2)/Z(G_\delta)\}^\wedge \end{array}$$

(Here the left vertical map was defined in (a), the right one is induced by the inclusion $Z(G) \hookrightarrow Z(G_\delta)$, and the horizontal maps are the homomorphisms (1.2.3) from 1.2.8.)

Proof. (a) Embed T into G, G_δ, H and H' by a, a_δ, c and c' , respectively. Then the set of roots of $(\widehat{G}_\delta, \widehat{T})$ (resp. of $(\widehat{H}, \widehat{T})$, resp. of $(\widehat{H}', \widehat{T})$) is the set of those roots $\hat{\alpha}$ of $(\widehat{G}, \widehat{T})$ such that $\alpha(\delta) = 1$ (resp. $\hat{\alpha}(\kappa) = 1$, resp. $\alpha(\delta) = 1$ and $\hat{\alpha}(\kappa) = 1$). In particular, the set of roots of $(\widehat{H}', \widehat{T})$ is canonically a subset of those of $(\widehat{H}, \widehat{T})$. This gives us the required embeddings $W(H') \hookrightarrow W(H)$ and $Z(\widehat{H}) \hookrightarrow Z(\widehat{H}')$. The last assertion follows from the fact that both \bar{s} and \bar{s}' are the classes of κ .

(b) Choose an embedding $b' : S \hookrightarrow H'$ from $[b']$, and identify S with $b'(S) \subset H'$ and T with $c'(T) \subset H'$. Choose $g \in H'(\overline{E})$ such that $gSg^{-1} = T$. Then $\text{Int } g$ defines an isomorphism $S_{\overline{E}} \xrightarrow{\sim} T_{\overline{E}}$. Let $b : S_{\overline{E}} \hookrightarrow H$ be the composition $c \circ \text{Int } g$. We claim that the $H(\overline{E})$ -conjugacy class $[b]$ of b is Γ -invariant and independent of the choices of g and b' .

If $g' \in H'(\overline{E})$ is another element such that $g'Sg'^{-1} = T$, then $g^{-1}g' \in \text{Norm}_{H'}(S)$, and $b' := c \circ \text{Int } g'$ equals $b \circ \text{Int}(g^{-1}g')$. But $\text{Int}(g^{-1}g') : S_{\overline{E}} \rightarrow S_{\overline{E}}$ is induced by an element of $W(H') \subset W(H)$, therefore b' is conjugate to b . Thus $[b]$ is independent of the choice of g . For each $\sigma \in \Gamma$, we have ${}^\sigma b = c \circ \text{Int}({}^\sigma g)$ and ${}^\sigma gS({}^\sigma g)^{-1} = T$. Hence

from the shown above, ${}^\sigma b$ is conjugate to b . Finally, if b' is replaced by $\text{Int } h \circ b'$ and g by gh^{-1} for some $h \in H'(\overline{E})$, then the resulting isomorphism $S_{\overline{E}} \xrightarrow{\sim} T_{\overline{E}}$ (and, therefore, b) do not change.

(c) Since the assertion is over \overline{E} , we can identify $S_{\overline{E}} \xrightarrow{\sim} T_{\overline{E}}$, as in (b), and thus replace $[b']$ by $[c']$ and $[b]$ by $[c]$. Now the first assertion follows from the fact that $\iota \circ [c']_{G_\delta} = \iota \circ [a_\delta] = [a] = [c]_G$, while the second one was the definition of the embedding $Z(\widehat{H}) \hookrightarrow Z(\widehat{H}')$.

(d) Mimicking the proof of (a) and (b), we see that there exists a Γ -equivariant embedding $Z(\widehat{H^2/Z(G)}) \hookrightarrow Z(\widehat{(H')^2/Z(G)})$ characterized by the following property: For each pair $[c'_i]$ of stable conjugacy classes of embeddings of maximal tori $S_i \hookrightarrow H'$ with the corresponding stable conjugacy classes $[c_i]$ of embeddings of maximal tori $S_i \hookrightarrow H$, the embedding $Z_{[c_1, c_2]} : Z(\widehat{H^2/Z(G)}) \hookrightarrow \{(S_1 \times S_2)/Z(G)\}^\wedge$ is the restriction of $Z_{[c'_1, c'_2]} : Z(\widehat{(H')^2/Z(G)}) \hookrightarrow \{(S_1 \times S_2)/Z(G)\}^\wedge$. Then our diagram extends to the following diagram

$$\begin{array}{ccccc}
Z(\widehat{H}) & \xrightarrow{\mu_H} & Z(\widehat{H^2/Z(G)}) & \xrightarrow{Z_{[b_1, b_2]}} & \{(T_1 \times T_2)/Z(G)\}^\wedge \\
\downarrow & & \downarrow & & \parallel \\
Z(\widehat{H'}) & \xrightarrow{\mu_{H'}} & Z(\widehat{(H')^2/Z(G)}) & \xrightarrow{Z_{[b'_1, b'_2]}} & \{(T_1 \times T_2)/Z(G)\}^\wedge \\
\parallel & & \uparrow & & \uparrow \\
Z(\widehat{H'}) & \xrightarrow{\mu_{H'}} & Z(\widehat{(H')^2/Z(G_\delta)}) & \xrightarrow{Z_{[b'_1, b'_2]}} & \{(T_1 \times T_2)/Z(G_\delta)\}^\wedge
\end{array}$$

It remains to show that each inner square of the diagram is commutative. The commutativity of the top right square follows from the characterization of the embedding $Z(\widehat{H^2/Z(G)}) \hookrightarrow Z(\widehat{(H')^2/Z(G)})$. The commutativity of the top left square follows from the characterization of the vertical maps and the fact that both μ_H and $\mu_{H'}$ are restrictions of $\mu_T : \widehat{T} \hookrightarrow \widehat{T^2/Z(G)}$. The commutativity of the two bottom squares is clear. \square

Corollary 2.4.3. *Let $[b'_1]$ and $[b'_2]$ be stable conjugacy classes of embeddings of maximal tori $S \hookrightarrow H'$ such that $[b'_1]_{G_\delta} = [b'_2]_{G_\delta}$, and let $[b_1]$ and $[b_2]$ be the corresponding stable conjugacy classes of embeddings $S \hookrightarrow H$.*

Then $[b_1]_G = [b_2]_G$, and the image of $\kappa\left(\frac{[b'_1]}{[b'_2]}\right)(\overline{s}) \in \pi_0(\widehat{S/Z(G_\delta)})^\Gamma$ in $\pi_0(\widehat{S/Z(G)})^\Gamma$ equals $\kappa\left(\frac{[b_1]}{[b_2]}\right)(\overline{s})$.

Proof. The first assertion follows from Lemma 2.4.2 (c). For the second one, choose a representative $\check{s} \in \pi_0(Z(\widehat{H})^\Gamma)$ of \overline{s} , and let $\check{s}' \in \pi_0(Z(\widehat{H}')^\Gamma)$ be the image of \check{s} . Then $\kappa(\frac{[b_1]}{[b_2]})(\overline{s}) = \nu_S(\iota([b_1], [b_2])(\check{s}))$ and $\kappa(\frac{[b'_1]}{[b'_2]})(\overline{s}') = \nu_S(\iota([b'_1], [b'_2])(\check{s}'))$. So the assertion follows from Lemma 2.4.2 (d). \square

2.4.4. Let $\varphi : G \rightarrow G'$ be an inner twisting such that $\delta' := \varphi(\delta)$ belongs to $G'(E)$. For each $i = 1, 2$, let $(a_\delta)_i : T_i \hookrightarrow G_\delta$ and $(a'_\delta)_i : T_i \hookrightarrow G'_\delta$ be stably conjugate embeddings of maximal tori, and let $b'_i : T_i \hookrightarrow H'$ be an embedding of a maximal torus compatible with $(a_\delta)_i$.

Let $\iota' : G'_\delta \hookrightarrow G'$ be the natural embedding, and for each $i = 1, 2$, set $a_i := \iota \circ (a_\delta)_i : T_i \hookrightarrow G$ and $a'_i := \iota' \circ (a'_\delta)_i : T_i \hookrightarrow G'$, and denote by $[b_i]$ the stable conjugacy class of embeddings of maximal tori $T_i \hookrightarrow H'$ corresponding to $[b'_i]$ (and compatible with a_i by Lemma 2.4.2 (b)).

Lemma 2.4.5. (a) *The image of $\overline{\text{inv}}((a_1, a'_1); (a_2, a'_2)) \in H^1(E, (T_1 \times T_2)/Z(G))$ in $H^1(E, (T_1 \times T_2)/Z(G_\delta))$ equals $\overline{\text{inv}}((a_\delta)_1, (a'_\delta)_1); ((a_\delta)_2, (a'_\delta)_2)$.*

(b) *We have an equality $\left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle_{\mathcal{E}} = \left\langle \frac{(a_\delta)_1, (a'_\delta)_1; [b'_1]}{(a_\delta)_2, (a'_\delta)_2; [b'_2]} \right\rangle_{\mathcal{E}'}$.*

Proof. (a) Follows immediately from the definition of the invariant.

(b) Let \check{s} and \check{s}' be as in the proof of Corollary 2.4.3. Then by Lemma 2.4.2 (a) and (d), we obtain that $\kappa([b_1], [b_2])(\check{s}) \in \pi_0(\widehat{\{(T_1 \times T_2)/Z(G)\}}^\Gamma)$ is the image of $\kappa([b'_1], [b'_2])(\check{s}') \in \pi_0(\widehat{\{(T_1 \times T_2)/Z(G_\delta)\}}^\Gamma)$. Now the assertion follows from (a) and the functoriality of the Tate–Nakayama duality. \square

2.4.6. Let $d_\delta : S \hookrightarrow G_\delta$ be an embedding of a maximal torus, $\overline{\xi}$ an element of $\pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)$, $d := \iota \circ d_\delta : S \hookrightarrow G$, $\overline{\xi}_\delta \in \pi_0(\widehat{S}^\Gamma/Z(\widehat{G}_\delta)^\Gamma)$ the class of $\overline{\xi}$, and $\varphi : G \rightarrow G'$ an $(\mathcal{E}, [d], \overline{\xi})$ -admissible inner twisting.

Lemma 2.4.7. *Assume that there exists $[b'] \in \Pi_{\mathcal{E}'}^{-1}([d_\delta], \overline{\xi}_\delta)$ such that the corresponding stable conjugacy class $[b]$ of embeddings $S \hookrightarrow H$ satisfies $\Pi_{\mathcal{E}}([b]) = ([d], \overline{\xi})$.*

If there exists a stably conjugate embedding $d' : S \hookrightarrow G'$ of d , then there exists a stable conjugate d' of d for which $G'_{d'(d^{-1}(\delta))}$ is an $(\mathcal{E}', [d_\delta], \overline{\xi}_\delta)$ -admissible inner form of G_δ .

Proof. For shortness, we will denote $Z(\mathcal{E}, [d], \overline{\xi}) \subset Z(\widehat{G^{\text{ad}}})^\Gamma$ by Z and $Z(\mathcal{E}', [d_\delta], \overline{\xi}_\delta) \subset Z(\widehat{G_\delta^{\text{ad}}})^\Gamma$ by Z_δ . Embedding $d_\delta : S \hookrightarrow G_\delta \subset G$ induce homomorphisms

$$H^1(E, (G_\delta)^{\text{ad}}) \xleftarrow{g} H^1(E, S/Z(G)) \xrightarrow{f} H^1(E, G^{\text{ad}}),$$

hence dual homomorphisms $Z(\widehat{(G_\delta)^{\text{ad}}})^\Gamma \xrightarrow{g^D} \pi_0(\widehat{S/Z(G)})^\Gamma \xleftarrow{f^D} Z(\widehat{G^{\text{ad}}})^\Gamma$.

First we claim that the assertion of the lemma is equivalent to the inclusion

$$(2.4.1) \quad g^D(Z_\delta) \cap \text{Im } f^D \subset f^D(Z).$$

Indeed, put $x := \text{inv}(G, G')$. By our assumptions, $x \in Z^\perp \cap \text{Im } f$, and the lemma is equivalent to the assertion that there exists $y \in f^{-1}(x)$ such that $g(y) \in (Z_\delta)^\perp$. Equivalently, we have to show that $f^{-1}(x) \cap g^{-1}((Z_\delta)^\perp) \neq \emptyset$. Since $g^{-1}((Z_\delta)^\perp) = [g^D(Z_\delta)]^\perp$, we have to check that x belongs to $f([g^D(Z_\delta)]^\perp) = [(f^D)^{-1}(g^D(Z_\delta))]^\perp$. In other words, the lemma asserts that $\text{Im } f \cap Z^\perp \subset [(f^D)^{-1}(g^D(Z_\delta))]^\perp$, or by duality that $(f^D)^{-1}(g^D(Z_\delta)) \subset Z + \text{Ker } f^D$. But the last inclusion is equivalent to (2.4.1).

To show (2.4.1), take any element $y \in g^D(Z_\delta) \cap \text{Im } f^D \subset \pi_0(\widehat{S/Z(G)})^\Gamma$. Note that g^D factors through $Z_{[\widehat{d_\delta}]} : Z((\widehat{G_\delta})^{\text{ad}})^\Gamma \rightarrow \pi_0(\widehat{S/Z(G_\delta)})^\Gamma$. Therefore it follows from Corollary 1.4.2 that there exists $[b'_1] \in \Pi_{\mathcal{E}'}^{-1}([d_\delta], \bar{\xi}_\delta)$ such that y is the image of $\kappa\left(\frac{[b'_1]}{[b]}\right)(\bar{s}') \in \pi_0(\widehat{S/Z(G_\delta)})^\Gamma$. Let $[b_1]$ be the stable conjugacy class of embeddings $S \hookrightarrow H$ corresponding to $[b'_1]$. Then $[b_1]_G = [b]_G$ and $y = \kappa\left(\frac{[b_1]}{[b]}\right)(\bar{s})$ (by Corollary 2.4.3). Since $y \in \text{Im } f^D$, the image of $\kappa\left(\frac{[b_1]}{[b]}\right)(\bar{s})$ in $\pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)$ is trivial. Since this image equals $\bar{\kappa}_{[b_1]}/\bar{\kappa}_{[b]}$ (use Lemma 1.2.9 (a)), we conclude that $\Pi_{\mathcal{E}}([b_1]) = \Pi_{\mathcal{E}}([b]) = ([d], \bar{\xi})$. Therefore $y = \kappa\left(\frac{[b_1]}{[b]}\right)(\bar{s})$ belongs to $f^D(Z)$, as claimed. \square

2.5. Preparation for the proof of the Main Theorem.

Lemma 2.5.1. *Let $\pi : \tilde{G} \rightarrow G$ be a quasi-isogeny such that \tilde{G} splits over E^{nr} , $\tilde{a}_0 : \tilde{T} \hookrightarrow \tilde{G}$ the lift of $a_0 : T \hookrightarrow G$, and $\tilde{\theta}$ the composition $\tilde{T}(E) \rightarrow T(E) \xrightarrow{\theta} \mathbb{C}^\times$.*

For each stable conjugate a of a_0 , the representation $\pi_{a,\theta} \circ \pi$ of $\tilde{G}(E)$ is isomorphic to the direct sum $\sum_{\tilde{a}} \pi_{\tilde{a},\tilde{\theta}}$, taken over the set of all conjugacy classes of embeddings $\tilde{a} : \tilde{T} \hookrightarrow \tilde{G}$ such that $\pi \circ \tilde{a} : T \hookrightarrow G$ is conjugate to a .

Proof. Observe first that for each quasi-isogeny $\bar{\pi} : \tilde{L} \rightarrow L_a$, the representation $\rho_{\bar{a},\bar{\theta}} \circ \bar{\pi}$ of $\tilde{L}(\mathbb{F}_q)$ is isomorphic to the Deligne–Lusztig representation $\rho_{\tilde{a},\tilde{\theta}}$, where $\tilde{a} : \tilde{T} \hookrightarrow \tilde{L}$ is the lift of \bar{a} , and $\tilde{\theta}$ is the composition $\tilde{T}(\mathbb{F}_q) \rightarrow \bar{T}(\mathbb{F}_q) \xrightarrow{\bar{\theta}} \mathbb{C}^\times$.

For each \tilde{a} as in the lemma, denote by $\pi_{\tilde{a}} : \tilde{G}_{\tilde{a}} \rightarrow \tilde{G}_a$ the restriction of π . Then by the above observation, the representation $\rho_{a,\theta} \circ \pi_{\tilde{a}}$ is isomorphic to $\rho_{\tilde{a},\tilde{\theta}}$. It follows that each $\pi_{\tilde{a},\tilde{\theta}}$ is a subrepresentation of $\pi_{a,\theta} \circ \pi$.

Since conjugacy classes of the \tilde{a} 's are naturally identified with the double coset $\pi(\tilde{G}(E)) \backslash G(E)/a(T(E))$, while the set of irreducible factors of $\pi_{a,\theta} \circ \pi$ is naturally

identified with $\pi(\widetilde{G}(E)) \backslash G(E) / \widetilde{G}_a$, it remains to check that these two double cosets coincide.

By Corollary 2.2.7, we have $\widetilde{G}_a = a(T(E))G_a$, therefore it will suffice to show that $G_a \subset \pi(\widetilde{G}(E))a(T(\mathcal{O}))$. But this inclusion follows from Corollary 1.8.18. \square

Corollary 2.5.2. *In the notation of Lemma 2.5.1, let $\tilde{\kappa} \in \widehat{\widetilde{T}}^\Gamma / Z(\widehat{\widetilde{G}})^\Gamma$ be the image of $\kappa \in \widehat{T}^\Gamma / Z(\widehat{G})^\Gamma$. Then $\pi^*(\chi_{a_0, \kappa, \theta}) = \chi_{\tilde{a}_0, \tilde{\kappa}, \tilde{\theta}}$.*

Proof. Since each \tilde{a} as in the lemma satisfies $\langle \text{inv}(\tilde{a}_0, \tilde{a}), \tilde{\kappa} \rangle = \langle \text{inv}(a_0, a), \kappa \rangle$, the assertion follows from the lemma. \square

Lemma 2.5.3. *It will suffice to prove Theorem 2.1.6 under the assumption that the derived group of G is simply connected.*

Proof. Let G be an arbitrary group satisfying the assumptions of Theorem 2.1.6. Since G splits over E^{nr} , there exists a surjective quasi-isogeny $\pi : \widetilde{G} \rightarrow G$ such that $\widetilde{G}^{\text{der}} = \widetilde{G}^{\text{sc}}$, and $\text{Ker } \pi$ is an induced torus splitting over E^{nr} (use [MS, Prop. 3.1]). (Such a quasi-isogeny Kottwitz calls z -extension.) Let $\tilde{a}_0 : \widetilde{T} \hookrightarrow \widetilde{G}$ be the lift of a_0 . Then \widetilde{T} splits over E^{nr} , hence \widetilde{G} satisfies all the assumptions of Theorem 2.1.6.

Let $\tilde{\kappa} \in \widehat{\widetilde{T}}^\Gamma / Z(\widehat{\widetilde{G}})^\Gamma$ be the image of κ , $\tilde{\theta}$ the composition $\widetilde{T}(E) \rightarrow T(E) \xrightarrow{\theta} \mathbb{C}^\times$, $\pi' : \widetilde{G}' \rightarrow G'$ the inner twist of π , induced by φ , and $\tilde{a}'_0 : \widetilde{T} \hookrightarrow \widetilde{G}'$ the lift of $a'_0 : T \hookrightarrow G'$. By the assumption, generalized functions $\chi_{\tilde{a}_0, \tilde{\kappa}, \tilde{\theta}}$ and $\chi_{\tilde{a}'_0, \tilde{\kappa}, \tilde{\theta}}$ are $(\tilde{a}_0, \tilde{a}'_0; \tilde{\kappa})$ -equivalent. Since $H^1(E, \text{Ker } \pi) = 0$, we get that $\pi(\widetilde{G}(E)) = G(E)$ and $\pi'(\widetilde{G}'(E)) = G'(E)$. Therefore it follows from Corollaries 2.5.2 and 1.7.15 that generalized functions $\chi_{a_0, \kappa, \theta}$ and $\chi_{a'_0, \kappa, \theta}$ are $(a_0, a'_0; \kappa)$ -equivalent. Thus by Corollary 1.6.13, they are $(a_0, a'_0; \kappa)$ -equivalent, as claimed. \square

From now on we will assume that $G^{\text{der}} = G^{\text{sc}}$.

Lemma 2.5.4. *Let G and a_0 be as in Theorem 2.1.6, $\delta \in G(E)$ an element of finite order prime to p , and $b_0 : T \hookrightarrow G_\delta$ an embedding stably conjugate to a_0 . Then*

- (a) *The conclusion of Theorem 2.2.3 holds for G_δ and b_0 .*
- (b) *The set of topologically unipotent elements of $G_\delta(E)$ equals $G_\delta(E)_{\text{tu}}$.*

Proof. (a) First of all, it follows from Lemma 2.3.8 (b) that G_δ splits over E^{nr} . Consider the canonical isogeny $\pi : G^{\text{sc}} \times Z(G)^0 \rightarrow G$. It induces the isogeny $\pi_\delta : (G^{\text{sc}})_\delta \times Z(G)^0 \rightarrow G_\delta$, where $(G^{\text{sc}})_\delta := \{g \in G^{\text{sc}} \mid \text{Int } \delta(g) = g\}$ is connected by [St, Thm. 8.1].

Since G satisfies property (vg) , the order of $Z(G^{\text{sc}})$ is prime to p . Therefore π and hence π_δ are of order prime to p . As in the proof of Lemma 2.5.3, we see (using

Lemma 1.8.17 (a) and Corollaries 2.5.2, 1.7.15 and 1.6.13) that we can replace G_δ by $(G^{\text{sc}})_\delta \times Z(G)^0$ and b_0 by its lift. Thus it will suffice to show that $(G^{\text{sc}})_\delta$ and the lift $b_0^{\text{sc}} : T^{\text{sc}} \hookrightarrow (G^{\text{sc}})_\delta$ of b_0 satisfy the assumptions of 2.2.1.

Since G satisfies property (vg), G^{sc} admits a quasi-logarithm $\Phi : G^{\text{sc}} \rightarrow \mathcal{G}^{\text{sc}}$ defined over \mathcal{O} , and \mathcal{G}^{sc} admits a non-degenerate over \mathcal{O} invariant pairing $\langle \cdot, \cdot \rangle$. As $\text{Lie}(G^{\text{sc}})_\delta = (\mathcal{G}^{\text{sc}})_\delta$, the quasi-logarithm Φ induces a quasi-logarithm Φ_δ for $(G^{\text{sc}})_\delta$, and $\langle \cdot, \cdot \rangle$ induces an invariant pairing $\langle \cdot, \cdot \rangle_\delta$ on $\text{Lie}(G^{\text{sc}})_\delta$. Furthermore, as δ is of finite order prime to p , we get that Φ_δ is defined over \mathcal{O} and $\langle \cdot, \cdot \rangle_\delta$ is non-degenerate over \mathcal{O} . Finally, since $b_0^{\text{sc}} : T^{\text{sc}} \hookrightarrow (G^{\text{sc}})_\delta$ is stably conjugate to $a_0^{\text{sc}} : T^{\text{sc}} \hookrightarrow G^{\text{sc}}$ and since G satisfies property (vg) $_{a_0}$, there exists a b_0^{sc} -strongly regular element of $\overline{T}^{\text{sc}}(\mathbb{F}_q)$, implying the last assumption of 2.2.1.

(b) The proof is a generalization of that of Corollary 2.3.3. Each topologically unipotent element $u \in G_\delta(E)$ stabilizes some point $x \in \mathcal{B}(G_\delta) \subset \mathcal{B}(G)$. Hence u belongs to $G_\delta(E) \cap G_{x,\text{tu}}$ (by Lemma 2.3.1 (b)). Therefore $\bar{u} \in L_x(\mathbb{F}_q)$ belongs to $(L_x)_{\bar{\delta}}^0 \cap \mathcal{U}(L_x)$ (by Lemma 2.3.8 (b)). By Lemma 2.5.5 below, \bar{u} belongs to $(L_x)_{\bar{\delta}}^0 \cap \mathcal{U}(L_x) = \mathcal{U}((L_x)_{\bar{\delta}}^0)$, hence u belongs to $(G_\delta)_{x,\text{tu}} \subset G_\delta(E)_{\text{tu}}$, as claimed. \square

Lemma 2.5.5. *Let L be a connected reductive group, and $s \in L$ a semisimple element. Then $\mathcal{U}(L)_s = \mathcal{U}(L) \cap L_s$ is contained in L_s^0 .*

Proof. Recall that the canonical homomorphism $\iota : L^{\text{sc}} \rightarrow L$ induces an L -equivariant isomorphism $\mathcal{U}(L^{\text{sc}}) \xrightarrow{\sim} \mathcal{U}(L)$, hence an isomorphism $\mathcal{U}(L^{\text{sc}})_s \xrightarrow{\sim} \mathcal{U}(L)_s$. Therefore $\mathcal{U}(L)_s = \iota(\mathcal{U}(L^{\text{sc}})_s)$ is contained in $\iota((L^{\text{sc}})_s)$. Since $(L^{\text{sc}})_s$ is connected (by [St, Thm. 8.1]), the latter group is contained in L_s^0 , as claimed. \square

Notation 2.5.6. (a) For a compact element $\gamma_0 \in G^{\text{sr}}(E)$ with a topological Jordan decomposition $\gamma_0 = \delta_0 u_0$, we say that $t \in T(E)$ is (G, a_0, γ_0) -relevant, if there exists an embedding $b_0 : T \hookrightarrow G_{\delta_0} \subset G$ stably conjugate to a_0 such that $b_0(t) = \delta_0$.

(b) Assume that $t \in T(E)$ is (G, a_0, γ_0) -relevant. Since $b_0(T) \subset G_{\delta_0}$ is elliptic, Kottwitz' theorem (see 1.5.1) implies that $H^1(E, T) \rightarrow H^1(E, G_{\delta_0})$ is surjective (compare the proof of Lemma 1.5.3). Hence for each $\delta \in G(E)$ stably conjugate to δ_0 there exists an embedding $b_{t,\delta} : T \hookrightarrow G_\delta \subset G$ stably conjugate to a_0 such that $b_{t,\delta}(t) = \delta$. Furthermore, $b_{t,\delta}$ is unique up to a stable conjugacy, and the endoscopic triple $\mathcal{E}_t := \mathcal{E}_{([b_{t,\delta}], \kappa)} = (H_t, [\eta_t], \bar{s}_t)$ of G_{δ_0} is independent of δ .

Lemma 2.5.7. *For each compact element $\gamma_0 \in G^{\text{sr}}(E)$ with topological Jordan decomposition $\gamma_0 = \delta_0 u_0$ and each $\bar{\xi} \in \pi_0(\widehat{G_{\gamma_0}}^\Gamma / Z(\widehat{G})^\Gamma)$, we have*

$$(2.5.1) \quad F_{a_0, \kappa, \theta}(\gamma_0, \bar{\xi}) = \sum_t \theta(t) \sum_\delta I_{t, \delta},$$

where

- (i) t runs over the set of (G, a_0, γ_0) -relevant elements of $T(E)$;
- (ii) δ runs over a set of representatives of the conjugacy classes in $G(E)$ stably conjugate to δ_0 , for which there exists a stably conjugate γ of γ_0 with topological Jordan decomposition $\gamma = \delta u$;
- (iii) $I_{t,\delta}$ equals

$$(2.5.2) \quad \langle \text{inv}(a_0, b_{t,\delta}), \kappa \rangle \langle \text{inv}(\gamma_0, \gamma), \bar{\xi} \rangle^{-1} F_{b_{t,\delta}, \kappa, \theta}(u, \bar{\xi})$$

for each $\gamma = \delta u$ as in (ii).

Proof. Recall that

$$F_{a_0, \kappa, \theta}(\gamma_0, \bar{\xi}) = e(G) \sum_a \sum_{\gamma} \langle \text{inv}(a_0, a), \kappa \rangle \langle \text{inv}(\gamma_0, \gamma), \bar{\xi} \rangle^{-1} F_{a, \theta}(\gamma),$$

where $a : T \hookrightarrow G$ and $\gamma \in G(E)$ run over sets of representatives of the conjugacy classes within the stable conjugacy classes of a_0 and γ_0 , respectively.

Using Proposition 2.3.7, we see that $F_{a_0, \kappa, \theta}(\gamma_0, \bar{\xi})$ equals the triple sum

$$(2.5.3) \quad \sum_{\gamma = \delta u} \sum_a \langle \text{inv}(a_0, a), \kappa \rangle \langle \text{inv}(\gamma_0, \gamma), \bar{\xi} \rangle^{-1} e(G_\delta) \sum_b \theta(b^{-1}(\delta)) F_{b, \theta}(u),$$

where γ and a are as above, and b runs over conjugacy classes of embeddings $T \hookrightarrow G_\delta$, which are conjugate to $a : T \hookrightarrow G$.

Then (2.5.3) can be rewritten in the form

$$(2.5.4) \quad \sum_{\gamma = \delta u} e(G_\delta) \sum_b \langle \text{inv}(a_0, b), \kappa \rangle \langle \text{inv}(\gamma_0, \gamma), \bar{\xi} \rangle^{-1} \theta(b^{-1}(\delta)) F_{b, \theta}(u),$$

where b runs over the set of conjugacy classes of embeddings $T \hookrightarrow G_\delta$, whose composition with the inclusion $G_\delta \hookrightarrow G$ is stably conjugate to a_0 .

Furthermore, each $t := b^{-1}(\delta) \in T(E)$, appearing in the sum, is (G, a_0, γ_0) -relevant, and the contribution of each such t to (2.5.4) is

$$(2.5.5) \quad \theta(t) \sum_{\gamma = \delta u} \langle \text{inv}(a_0, b_{t,\delta}), \kappa \rangle \langle \text{inv}(\gamma_0, \gamma), \bar{\xi} \rangle^{-1} e(G_\delta) \sum_b \langle \text{inv}(b_{t,\delta}, b), \kappa \rangle F_{b, \theta}(u),$$

where b runs over a set of representatives of conjugacy classes of embeddings $T \hookrightarrow G_\delta$ stably conjugate to $b_{t,\delta}$. But (2.5.5) coincides with the sum $\theta(t) \sum_{\delta} I_{t,\delta}$ as in the lemma. \square

2.5.8. We fix $\gamma_0 \in G^{\text{sr}}(E)$ and $\bar{\xi} \in \pi_0(\widehat{G_{\gamma_0}}^\Gamma / Z(\widehat{G})^\Gamma)$ such that $F_{a_0, \kappa, \theta}(\gamma_0, \bar{\xi}) \neq 0$, and we are going to show that $(\gamma_0, \bar{\xi})$ satisfies the conditions (i), (ii) of Theorem 2.1.10.

2.6. Proof of Theorem 2.1.10 (i).

2.6.1. Since the support of each $t_{a,\theta}$ is contained in \widetilde{G}_a , the assumption on γ_0 implies that there exists $z \in Z(G)(E)$ such that $z\gamma_0$ is compact. But $F_{a_0,\kappa,\theta}(z\gamma_0, \bar{\xi}) = \theta(z)F_{a_0,\kappa,\theta}(\gamma_0, \bar{\xi})$ for each $z \in Z(G)(E)$, therefore the assertions of Theorem 2.1.10 for γ_0 are equivalent to those for $z\gamma_0$. Hence we can and will assume that γ_0 is compact. In particular, Lemma 2.5.7 holds for γ_0 .

Every stably conjugate γ of γ_0 is compact as well, and we denote by $\gamma_0 = \delta_0 u_0$ and $\gamma = \delta u$ their topological Jordan decompositions. We also let $\bar{\xi}_{\delta_0}$ be the image of $\bar{\xi}$ in $\pi_0(\widehat{G_{\gamma_0}}^\Gamma / Z(\widehat{G_{\delta_0}})^\Gamma)$.

Notation 2.6.2. We will say that stable conjugates $\delta_1, \delta_2 \in G(E)$ of δ_0 are $(\gamma_0, \bar{\xi})$ -*equivalent*, if there exist stable conjugate γ_1 and γ_2 of γ_0 with topological Jordan decompositions $\gamma_i = \delta_i u_i$, and G_{δ_2} is an $(\mathcal{E}_t, [a_{\gamma_0}], \bar{\xi}_{\delta_0})$ -admissible inner form of G_{δ_1} . In this case we will write $\delta_1 \sim_{(\gamma_0, \bar{\xi})} \delta_2$.

2.6.3. Fix t which has a non-zero contribution to (2.5.1). Since the set of conjugacy classes of δ in Lemma 2.5.7 (ii) decomposes as a union of $(\gamma_0, \bar{\xi})$ -equivalent classes, we can replace γ_0 by a stably conjugate element, so that $\sum_{\delta \sim_{(\gamma_0, \bar{\xi})} \delta_0} I_{t,\delta} \neq 0$. Further replacing γ_0 , we can moreover assume that $I_{t,\delta_0} \neq 0$, thus $F_{b_{t,\delta_0},\kappa,\theta}(u_0, \bar{\xi}_{\delta_0}) \neq 0$.

We also fix embeddings of maximal tori $c : T \hookrightarrow H$ and $c' : T \hookrightarrow H_t$ such that $\Pi_{\mathcal{E}}([c]) = ([a_0], \bar{\kappa})$ and $\Pi_{\mathcal{E}_t}([c']) = ([b_{t,\delta_0}], \bar{\kappa}')$ (see 2.4.1 (b)). This enables us to apply the results of Subsection 2.4.

2.6.4. Since $G^{\text{der}} = G^{\text{sc}}$, we get that $u_0 \in G_{\delta_0}(E)_{\text{tu}}$ and the conclusion of Theorem 2.2.3 holds for G_{δ_0} and b_{t,δ_0} (see Lemma 2.5.4). Therefore as in Lemma 2.1.11, there exists an embedding $b' : G_{\gamma_0} = (G_{\delta_0})_{u_0} \hookrightarrow H_t$ such that $\Pi_{\mathcal{E}_t}([b']) = ([a_{\gamma_0}], \bar{\xi}_{\delta_0})$, where $a_{\gamma_0} : G_{\gamma_0} \hookrightarrow G_{\delta_0}$ is the natural inclusion.

Let $[b]$ be the stable conjugacy class of embeddings $G_{\gamma_0} \hookrightarrow H$ corresponding to $[b']$ (see Lemma 2.4.2 (b)), and put $\bar{\xi}_{[b]} := \bar{\kappa}_{[b]} \in \pi_0(\widehat{G_{\gamma_0}}^\Gamma / Z(\widehat{G})^\Gamma)$.

To prove the assertion (i) of Theorem 2.1.10, it will suffice to show the existence of $[b'] \in \Pi_{\mathcal{E}_t}^{-1}([a_{\gamma_0}], \bar{\xi}_{\delta_0})$ such that $\bar{\xi}_{[b']} = \bar{\xi}$.

2.6.5. For each $[b'] \in \Pi_{\mathcal{E}_t}^{-1}([a_{\gamma_0}], \bar{\xi}_{\delta_0})$, we denote by $z_{[b']}$ the image of the quotient $\bar{\xi}_{[b]}/\bar{\xi} \in \pi_0(\widehat{G_{\gamma_0}}^\Gamma / Z(\widehat{G})^\Gamma)$ in $\pi_0(\widehat{G_{\gamma_0}}^\Gamma / Z(\widehat{G})^\Gamma Z(\mathcal{E}_t, [a_{u_0}], \bar{\xi}_{\delta_0}))$, where $Z(\mathcal{E}_t, [a_{u_0}], \bar{\xi}_{\delta_0}) \subset Z((\widehat{G_{\delta_0}})^{\text{ad}})^\Gamma$ is mapped into $\widehat{G_{\gamma_0}}^\Gamma$ via the homomorphism $Z((\widehat{G_{\delta_0}})^{\text{ad}}) \rightarrow G_{\gamma_0}/Z(G_{\delta_0}) \rightarrow \widehat{G_{\gamma_0}}^\Gamma$.

We claim that $z_{[b']}$ does not depend on $[b']$. Indeed, for each $[b'_1], [b'_2] \in \Pi_{\mathcal{E}_t}^{-1}([a_{\gamma_0}], \bar{\xi}_{\delta_0})$, the quotient $\bar{\xi}_{[b'_1]}/\bar{\xi}_{[b'_2]}$ is the image of $\kappa\left(\frac{[b'_1]}{[b'_2]}\right)(\bar{s}_t) \in \pi_0(\widehat{G_{\gamma_0}/Z(G_{\delta_0})}^\Gamma)$ (by Corollary 2.4.3 and Lemma 1.2.9 (a)). Thus $\bar{\xi}_{[b'_1]}/\bar{\xi}_{[b'_2]}$ belongs to the image of $Z(\mathcal{E}_t, [a_{u_0}], \bar{\xi}_{\delta_0})$. It follows that $z := z_{[b']}$ is independent of $[b']$, as claimed.

2.6.6. Our next goal is to show that $z = 1$. By definition, z belongs to the image

$$(2.6.1) \quad \text{Im} \left[\pi_0(\widehat{Z(G_{\delta_0})}^\Gamma / Z(\widehat{G})^\Gamma) \rightarrow \pi_0(\widehat{G_{\gamma_0}}^\Gamma / Z(\widehat{G})^\Gamma Z(\mathcal{E}_t, [a_{u_0}], \bar{\xi}_{\delta_0})) \right].$$

Denote by $V \subset \text{Ker}[H^1(E, G_{\delta_0}) \rightarrow H^1(E, G)]$ the intersection of the image of $\text{Ker}[H^1(E, G_{\gamma_0}) \rightarrow H^1(E, G)]$ and the preimage of $Z(\mathcal{E}_t, [a_{u_0}], \bar{\xi}_{\delta_0})^\perp \subset H^1(E, (G_{\delta_0})^{\text{ad}})$. Then for a stable conjugate $\delta \in G(E)$ of δ_0 , we have $\delta \sim_{(\gamma_0, \bar{\xi})} \delta_0$ if and only if the invariant $\text{inv}(\delta_0, \delta) \in \text{Ker}[H^1(E, G_{\delta_0}) \rightarrow H^1(E, G)]$ lies in V .

By Kottwitz' theorem (1.5.1), the dual group V^D of V is naturally identified with the group (2.6.1). In particular, z belongs to V^D . Therefore for each $\delta \sim_{(\gamma_0, \bar{\xi})} \delta_0$ one can form a pairing $\langle \text{inv}(\delta_0, \delta), z \rangle$. Explicitly,

$$(2.6.2) \quad \langle \text{inv}(\delta_0, \delta), z \rangle = \langle \text{inv}(\gamma_0, \gamma), \tilde{z} \rangle$$

for every stably conjugate γ of γ_0 with topological Jordan decomposition $\gamma = \delta u$, and every representative $\tilde{z} \in \pi_0(\widehat{G_{\gamma_0}}^\Gamma / Z(\widehat{G})^\Gamma)$ of z .

Claim 2.6.7. *For each $\delta \sim_{(\gamma_0, \bar{\xi})} \delta_0$, we have $I_{t, \delta} = \langle \text{inv}(\delta_0, \delta), z \rangle I_{t, \delta_0}$.*

Proof. Let $\gamma \in G(E)$ be a stable conjugate of γ_0 with topological Jordan decomposition $\gamma = \delta u$. Then $u \in G_\delta(E)$ is a stable conjugate of $u_0 \in G_{\delta_0}(E)$. Since by assumption $F_{b_{t, \delta_0}, \kappa, \theta}(u_0, \bar{\xi}_{\delta_0}) \neq 0$ and $u_0 \in G_{\delta_0}(E)$ is topologically unipotent, we conclude from Lemma 2.5.4 (as in Lemma 2.1.11) that

$$F_{b_{t, \delta}, \kappa, \theta}(u, \bar{\xi}_{\delta_0}) = \left\langle \frac{u_0, u; \bar{\xi}_{\delta_0}}{b_{t, \delta_0}, b_{t, \delta}; \kappa} \right\rangle_{\mathcal{E}_t} F_{b_{t, \delta_0}, \kappa, \theta}(u_0, \bar{\xi}_{\delta_0}).$$

Hence the quotient $I_{t, \delta}/I_{t, \delta_0}$ equals

$$\left\langle \frac{u_0, u; \bar{\xi}_{\delta_0}}{b_{t, \delta_0}, b_{t, \delta}; \kappa} \right\rangle_{\mathcal{E}_t} \langle \text{inv}(b_{t, \delta_0}, b_{t, \delta}), \kappa \rangle \langle \text{inv}(\gamma_0, \gamma), \bar{\xi} \rangle^{-1}.$$

Thus our claim is equivalent to the equality

$$(2.6.3) \quad \left\langle \frac{u_0, u; \bar{\xi}_{\delta_0}}{b_{t, \delta_0}, b_{t, \delta}; \kappa} \right\rangle_{\mathcal{E}_t} = \langle \text{inv}(\delta_0, \delta), z \rangle \langle \text{inv}(\gamma_0, \gamma), \bar{\xi} \rangle \langle \text{inv}(b_{t, \delta_0}, b_{t, \delta}), \kappa \rangle^{-1}.$$

The left hand side of (2.6.3) equals $\left\langle \frac{a_{u_0}, a_u; [b']}{b_{t, \delta_0}, b_{t, \delta}; [c']} \right\rangle_{\mathcal{E}_t}$ for each $[b'] \in \Pi_{\mathcal{E}_t}^{-1}([a_{\gamma_0}], \bar{\xi}_{\delta_0})$. By Lemma 2.4.5 (b) and Lemma 1.5.7 (d), it therefore equals

$$\left\langle \frac{a_{\gamma_0}, a_{\gamma}; [b]}{b_{t, \delta_0}, b_{t, \delta}; [c]} \right\rangle_{\mathcal{E}} = \langle \text{inv}(\gamma_0, \gamma), \bar{\kappa}_{[b]} \rangle \langle \text{inv}(b_{t, \delta_0}, b_{t, \delta}), \bar{\kappa}_{[c]} \rangle^{-1}.$$

But $\bar{\kappa}_{[c]} = \kappa$, $\bar{\kappa}_{[b]} = (\bar{\xi}_{[b']}/\bar{\xi})\bar{\xi}$ and $\bar{\xi}_{[b']}/\bar{\xi}$ is a representative of z . Therefore equality (2.6.3) follows from (2.6.2). \square

2.6.8. By Claim 2.6.7, the sum $\sum_{\delta \sim_{(\gamma_0, \bar{\xi})} \delta_0} I_{t, \delta_0}$ equals $I_{t, \delta_0}(\sum_{v \in V} \langle v, z \rangle)$. It follows that $\sum_{v \in V} \langle v, z \rangle \neq 0$, hence $z = 1$.

Choose now an arbitrary $[b'] \in \Pi_{\mathcal{E}_t}^{-1}([a_{\gamma_0}], \bar{\xi}_{\delta_0})$. Since $z = 1$, the quotient $\bar{\xi}_{[b']}/\bar{\xi} \in \pi_0(\widehat{G_{\gamma_0}}^{\Gamma}/Z(\widehat{G})^{\Gamma})$ lies in the image of $Z(\mathcal{E}_t, [a_{u_0}], \bar{\xi}_{\delta_0})$. Hence by Corollary 1.4.2, there exists $[b_1] \in \Pi_{\mathcal{E}_t}^{-1}([a_{\gamma_0}], \bar{\xi}_{\delta_0})$ such that $\bar{\xi}_{[b']}/\bar{\xi}$ equals the image of $\kappa\left(\frac{[b']}{[b_1]}\right)(\bar{s}_t)$. Since by Corollary 2.4.3 and Lemma 1.2.9 (a), the image of $\kappa\left(\frac{[b']}{[b_1]}\right)(\bar{s}_t)$ in $\pi_0(\widehat{G_{\gamma_0}}^{\Gamma}/Z(\widehat{G})^{\Gamma})$ equals $\bar{\xi}_{[b']}/\bar{\xi}_{[b_1]}$, we get that $\bar{\xi}_{[b_1]} = \bar{\xi}$, completing the proof of (i).

2.7. Proof of Theorem 2.1.10 (ii).

2.7.1. Let $\gamma'_0 \in G'(E)$ be a stable conjugate of γ_0 . Since γ_0 is compact, so is γ'_0 , and we denote by $\gamma'_0 = \delta'_0 u'_0$ its topological Jordan decomposition. By Lemma 2.5.7, we can write $F_{a'_0, \kappa, \theta}(\gamma'_0, \bar{\xi})$ in the form

$$F_{a'_0, \kappa, \theta}(\gamma'_0, \bar{\xi}) = \sum_{t'} \theta(t') \sum_{\delta'} I_{t', \delta'},$$

where t' , δ' and $I_{t', \delta'}$ have the same meaning as in Lemma 2.5.7.

First we claim that an element $t \in T(E)$ is (G, a_0, γ_0) -relevant if and only if it is (G', a'_0, γ'_0) -relevant. Indeed, assume that t is (G, a_0, γ_0) -relevant, and let $b_0 : T \hookrightarrow G_{\delta_0} \subset G$ be the corresponding embedding. Since $T/Z(G)$ is anisotropic, Kottwitz' theorem implies that the map $H^1(E, T/Z(G)) \rightarrow H^1(E, G_{\delta_0}/Z(G))$ is surjective (compare the proof of Lemma 1.5.3). Hence there exists a stable conjugate $b'_0 : T \hookrightarrow G'_{\delta'_0} \subset G'$ of b_0 such that $b'_0(t) = \delta'_0$, thus t is (G', a'_0, γ'_0) -relevant.

Therefore it will suffice to show that for each (G, a_0, γ_0) -relevant t , we have

$$(2.7.1) \quad \sum_{\delta'} I_{t, \delta'} = \left\langle \frac{\gamma_0, \gamma'_0; \bar{\xi}}{a_0, a'_0, \kappa} \right\rangle_{\mathcal{E}} \sum_{\delta} I_{t, \delta}.$$

2.7.2. Fix (G, a_0, γ_0) -relevant t . Generalizing Notation 2.6.2, we will say that stable conjugates $\delta \in G(E)$ and $\delta' \in G'(E)$ of δ_0 are $(\gamma_0, \bar{\xi})$ -equivalent (and will

write $\delta \sim_{(\gamma_0, \bar{\xi})} \delta'$, if there exist stable conjugates $\gamma \in G(E)$ and $\gamma' \in G'(E)$ of γ_0 with topological Jordan decompositions $\gamma = \delta u$ and $\gamma' = \delta' u'$ such that $G'_{\delta'}$ is an $(\mathcal{E}_t, [a_{u_0}], \bar{\xi}_{\delta_0})$ -admissible inner form of G_{δ} .

Assume that $\varphi : G \rightarrow G'$ is $(\mathcal{E}, [a_{\gamma_0}], \bar{\xi})$ -admissible. We claim that for every stable conjugate $\gamma \in G(E)$ of γ_0 with topological Jordan decomposition $\gamma = \delta u$, there exists a stable conjugate $\gamma' \in G'(E)$ of γ'_0 with topological Jordan decomposition $\gamma' = \delta' u'$ such that $\delta \sim_{(\gamma_0, \bar{\xi})} \delta'$.

Indeed, we have shown in Subsection 2.6 that there exists $[b'] \in \Pi_{\mathcal{E}_t}^{-1}([a_{\gamma_0}], \bar{\xi}_{\delta_0})$ such that the corresponding stable conjugacy class $[b]$ of embeddings $G_{\gamma_0} \hookrightarrow H$ satisfies $\Pi_{\mathcal{E}}([b]) = ([a_{\gamma_0}], \bar{\xi})$. Since the inclusion $a_{\gamma} : G_{\gamma} \hookrightarrow G$ has a stable conjugate $a_{\gamma'_0} : G_{\gamma'} \cong G'_{\gamma'_0} \hookrightarrow G'$, the assertion follows from Lemma 2.4.7.

Therefore equality (2.7.1) follows from the following generalization of Claim 2.6.7.

Claim 2.7.3. *For each $\delta' \sim_{(\gamma_0, \bar{\xi})} \delta$, we have $I_{t, \delta'} = \left\langle \frac{\gamma_0, \gamma'_0; \bar{\xi}}{a_0, a'_0; \kappa} \right\rangle_{\mathcal{E}} I_{t, \delta}$.*

Proof. The proof is very similar to that of Claim 2.6.7. By Theorem 2.2.3 for the inner twisting $G_{\delta} \rightarrow G'_{\delta'}$ (use Lemma 2.5.4), we have

$$F_{b_{t, \delta'}, \kappa, \theta}(u', \bar{\xi}_{\delta_0}) = \left\langle \frac{u, u'; \bar{\xi}_{\delta_0}}{b_{t, \delta}, b_{t, \delta'}; \kappa} \right\rangle_{\mathcal{E}_t} F_{b_{t, \delta}, \kappa, \theta}(u, \bar{\xi}_{\delta_0}).$$

Hence the quotient $I_{t, \delta'}/I_{t, \delta}$ equals

$$\left\langle \frac{u, u'; \bar{\xi}_{\delta_0}}{b_{t, \delta}, b_{t, \delta'}; \kappa} \right\rangle_{\mathcal{E}_t} \langle \text{inv}(a'_0, b_{t, \delta'}), \kappa \rangle \langle \text{inv}(a_0, b_{t, \delta}), \kappa \rangle^{-1} \langle \text{inv}(\gamma'_0, \gamma'), \bar{\xi} \rangle^{-1} \langle \text{inv}(\gamma_0, \gamma), \bar{\xi} \rangle.$$

Thus we have to check that $\left\langle \frac{u, u'; \bar{\xi}_{\delta_0}}{b_{t, \delta}, b_{t, \delta'}; \kappa} \right\rangle_{\mathcal{E}_t} = \left\langle \frac{a_{\gamma}, a_{\gamma'}; [b']}{b_{t, \delta}, b_{t, \delta'}; [c']} \right\rangle_{\mathcal{E}_t}$ equals

$$\left\langle \frac{\gamma_0, \gamma'_0; \bar{\xi}}{a_0, a'_0; \kappa} \right\rangle_{\mathcal{E}} \langle \text{inv}(a'_0, b_{t, \delta'}), \kappa \rangle^{-1} \langle \text{inv}(a_0, b_{t, \delta}), \kappa \rangle \langle \text{inv}(\gamma'_0, \gamma'), \bar{\xi} \rangle \langle \text{inv}(\gamma_0, \gamma), \bar{\xi} \rangle^{-1}.$$

Since the latter expression equals $\left\langle \frac{a_{\gamma}, a_{\gamma'}; [b]}{b_{t, \delta}, b_{t, \delta'}; [c]} \right\rangle_{\mathcal{E}}$ (use Lemma 1.5.7 (a), (c)), the assertion follows from Lemma 2.4.5 (b). \square

This completes the proof of Theorems 2.1.10 and 2.1.6.

APPENDIX A. SPRINGER HYPOTHESIS

The goal of this appendix is to prove the following result, conjectured by Springer and playing a crucial role in Subsection 2.2. In the case of large characteristic this result was first proved in [Ka1].

Theorem A.1. *Let L be a reductive group over a finite field \mathbb{F}_q , $\Phi : L \rightarrow \mathcal{L}$ a quasi-logarithm (see Definition 1.8.1), $\langle \cdot, \cdot \rangle$ a non-degenerate invariant pairing on \mathcal{L} , $T \subset L$ a maximal torus, θ a character of $T(\mathbb{F}_q)$, ψ a character of \mathbb{F}_q , and t an element of $\mathcal{T}(\mathbb{F}_q) \cap \mathcal{L}^{\text{sr}}(\mathbb{F}_q)$.*

Denote by δ_t the characteristic function of the $\text{Ad } L(\mathbb{F}_q)$ -orbit of t , by $\mathcal{F}(\delta_t)$ its Fourier transform, and by $(-1)^{\text{rk}_{\mathbb{F}_q}(L) - \text{rk}_{\mathbb{F}_q}(T)} \rho_{T,\theta}$ the Deligne–Lusztig [DL] virtual representation of $L(\mathbb{F}_q)$ corresponding to T and θ .

For every unipotent $u \in L(\mathbb{F}_q)$, we have

$$\text{Tr } \rho_{T,\theta}(u) = q^{-\frac{1}{2} \dim(L/T)} \mathcal{F}(\delta_t)(\Phi(u)).$$

Notation A.2. For each Weil sheaf \mathcal{A} over a variety X over a finite field \mathbb{F}_q , we denote by $\text{Func}(\mathcal{A})$ the corresponding function on $X(\mathbb{F}_q)$.

Proof. The theorem is a consequence of the results of Lusztig ([Lu]) and Springer ([Sp]), and seems to be well-known to experts. Set $\mathcal{U} := \mathcal{U}(L) \subset L$ and $\mathcal{N} := \mathcal{N}(\mathcal{L}) \subset \mathcal{L}$. To carry out the proof, we will construct a Weil sheaf \mathcal{A} on $\mathcal{L} \times \mathcal{T}$ such that the restrictions \mathcal{A}_t to $\mathcal{L} \times \{t\} \cong \mathcal{L}$ are perverse for all $t \in \mathcal{T}(\mathbb{F}_q)$ and satisfy the following properties:

- (i) The restriction $\mathcal{A}_t|_{\mathcal{N}}$ is independent of t .
- (ii) If $t \in \mathcal{L}^{\text{sr}}(\mathbb{F}_q)$, then $\text{Func}(\mathcal{A}_t) = q^{-\frac{1}{2} \dim(L/T)} \mathcal{F}(\delta_t)$.
- (iii) $\text{Func}(\Phi^*(\mathcal{A}_0))|_{\mathcal{U}} = \text{Tr } \rho_{T,\theta}|_{\mathcal{U}}$.

The existence of such an \mathcal{A} implies the Theorem. Indeed, fix $u \in \mathcal{U}(\mathbb{F}_q)$. By (iii), $\text{Tr } \rho_{T,\theta}(u)$ equals $\text{Func}(\mathcal{A}_0)(\Phi(u))$. Since $\Phi(\mathcal{U}) \subset \mathcal{N}$ (see [BR, 9.1, 9.2]), we have $\Phi(u) \in \mathcal{N}$. Therefore the assertion follows from (i) and (ii).

A.3. Construction of \mathcal{A} . Let T' be the abstract Cartan subgroup of L , and $W \subset \text{Aut}(T')$ the Weyl group of L (see [DL, 1.1]). Denote by $\tilde{\mathcal{L}}$ the Springer resolution of \mathcal{L} classifying pairs (\mathcal{B}, x) , where $\mathcal{B} \subset \mathcal{L}$ is a Borel subalgebra and x is an element of \mathcal{B} . Consider the diagram

$$\mathcal{L} \times \mathcal{T}' \xleftarrow{\pi \times \text{Id}} \tilde{\mathcal{L}} \times \mathcal{T}' \xrightarrow{\alpha \times \text{Id}} \mathcal{T}' \times \mathcal{T}' \xrightarrow{\langle \cdot, \cdot \rangle'} \mathbb{A}^1,$$

where π and α send (\mathcal{B}, x) to x and $\bar{x} \in \mathcal{B}/[\mathcal{B}, \mathcal{B}] = \mathcal{T}'$, respectively, and $\langle \cdot, \cdot \rangle'$ is the form on \mathcal{T}' induced by $\langle \cdot, \cdot \rangle$. Put

$$\mathcal{A}' := (\pi \times \text{Id})_!(\alpha \times \text{Id})^* \langle \cdot, \cdot \rangle'^* (\mathcal{L}_\psi)[\dim L],$$

where \mathcal{L}_ψ is the Artin-Schreier local system on \mathbb{A}^1 corresponding to ψ .

To construct \mathcal{A} , we will show first that for every $w \in W$, there exists a canonical isomorphism $(\text{Id} \times w)^* \mathcal{A}' \xrightarrow{\sim} \mathcal{A}'$. Denote by upper index $(\cdot)^{\text{sr}}$ the restriction to (the preimage of) \mathcal{L}^{sr} . First we will show that $(\text{Id} \times w)^* \mathcal{A}^{\text{sr}}$ is canonically isomorphic to \mathcal{A}^{sr} . As $\pi^{\text{sr}} : \tilde{\mathcal{L}}^{\text{sr}} \rightarrow \mathcal{L}^{\text{sr}}$ is an unramified Galois covering with the Galois group W ,

the functor π_1^{sr} is isomorphic to $\pi_1^{\text{sr}} \circ w^*$. Since α^{sr} is W -equivariant, and $\langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle' \circ (w \times w)$, we have a canonical isomorphism of functors

$$\begin{aligned} (\text{Id} \times w)^*(\pi^{\text{sr}} \times \text{Id})_!(\alpha^{\text{sr}} \times \text{Id})^* \langle \cdot, \cdot \rangle'^* &\cong (\pi^{\text{sr}} \times \text{Id})_!(\text{Id} \times w)^*(\alpha^{\text{sr}} \times \text{Id})^* \langle \cdot, \cdot \rangle'^* \cong \\ &(\pi^{\text{sr}} \times \text{Id})_!(w \times w)^*(\alpha^{\text{sr}} \times \text{Id})^* \langle \cdot, \cdot \rangle'^* \cong (\pi^{\text{sr}} \times \text{Id})_!(\alpha^{\text{sr}} \times \text{Id})^* \langle \cdot, \cdot \rangle'^*, \end{aligned}$$

implying the isomorphism $(\text{Id} \times w)^* \mathcal{A}'^{\text{sr}} \xrightarrow{\sim} \mathcal{A}'^{\text{sr}}$. Since α and $\langle \cdot, \cdot \rangle'$ are smooth morphisms, while π is small, we see that $\mathcal{A}'[\dim T]$ is a semisimple perverse sheaf, which is the intermediate extension of $\mathcal{A}'^{\text{sr}}[\dim T]$. Thus the constructed above isomorphism $(\text{Id} \times w)^* \mathcal{A}'^{\text{sr}} \xrightarrow{\sim} \mathcal{A}'^{\text{sr}}$ uniquely extends to an isomorphism $(\text{Id} \times w)^* \mathcal{A}' \xrightarrow{\sim} \mathcal{A}'$.

Denote by $\text{Fr} : T' \rightarrow T'$ the geometric Frobenius morphism corresponding to the \mathbb{F}_q -structure of T' , and choose an isomorphism between T and T' over $\overline{\mathbb{F}_q}$. Then there exists $w \in W$ such that the \mathbb{F}_q -structure of T corresponds to the morphism $\text{Fr}_w := w \circ \text{Fr} : T' \rightarrow T'$. Denote by \mathcal{A} the Weil sheaf \mathcal{A} on $\mathcal{L} \times \mathcal{T}$, which is isomorphic to \mathcal{A}' over $\overline{\mathbb{F}_q}$, and the Weil structure corresponds to the composition $\text{Fr}^*(\text{Id} \times w)^* \mathcal{A}' \xrightarrow{\sim} \text{Fr}^* \mathcal{A}' \xrightarrow{\sim} \mathcal{A}'$, where the first isomorphism was constructed above, and the second one comes from the Weil structure of \mathcal{A}' .

It remains to show that \mathcal{A} satisfies properties (i)–(iii).

A.4. Proof of properties (i)–(iii).

(i) Put $\tilde{\mathcal{N}} := \pi^{-1}(\mathcal{N}) \subset \tilde{\mathcal{L}}$. Then $\alpha(\tilde{\mathcal{N}}) = 0$, hence $(\alpha \times \text{Id})^* \langle \cdot, \cdot \rangle'^* (\mathcal{L}_\psi)|_{\tilde{\mathcal{N}} \times T'} \cong \overline{\mathbb{Q}_l}$. This implies the assertion for \mathcal{A}' . To show the assertion for \mathcal{A} , notice that $\tilde{\mathcal{N}}$ is smooth of dimension $\dim(L/T)$, and the projection $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$ is semi-small. It follows that $\mathcal{A}'|_{\mathcal{N} \times \mathcal{T}}$ is a semisimple perverse sheaf. Therefore for each $t \in \mathcal{T}(\mathbb{F}_q)$ the restriction map $\text{Hom}(\text{Fr}_w^* \mathcal{A}'|_{\mathcal{N} \times \mathcal{T}}, \mathcal{A}'|_{\mathcal{N} \times \mathcal{T}}) \rightarrow \text{Hom}(\text{Fr}_w^* \mathcal{A}'|_{\mathcal{N} \times \{t\}}, \mathcal{A}'|_{\mathcal{N} \times \{t\}})$ is an isomorphism. Thus the Weil structure of $\mathcal{A}_t|_{\mathcal{N}}$ is independent of t as well.

(ii) Denote by IC_t the constant perverse sheaf $\overline{\mathbb{Q}_l}(\frac{\dim(L/T)}{2})[\dim(L/T)]$ on the orbit $\text{Ad } L(t) \subset \mathcal{L}$, and let $\mathcal{F}(\text{IC}_t)$ be the Fourier–Deligne transform of IC_t . As $\dim(L/T)$ is even, we get that $\text{Func}(\mathcal{F}(\text{IC}_t)) = q^{-\frac{\dim(L/T)}{2}} \mathcal{F}(\delta_t)$. Therefore the assertion follows from well-known equality $\mathcal{F}(\text{IC}_t) = \mathcal{A}_t$ (see for example [Sp]).

(iii) By [DL, Thm. 4.2], $\text{Tr } \rho_{T, \theta}|_{\mathcal{U}(\mathbb{F}_q)}$ does not depend on θ , hence we can assume that $\theta = 1$. It was proved by Lusztig (see [Lu, Thm. 1.14 (a) and Prop. 8.15] and compare [BP]) that there exists a perverse sheaf \mathcal{K}_T on L such that $\text{Tr } \rho_{T, 1} = \text{Func}(\mathcal{K}_T)$. More precisely, in [Lu, Thm. 1.14 (a)] Lusztig showed the corresponding result for general character sheaves if q is sufficiently large, while by [Lu, Prop. 8.15] the restriction on q is unnecessary in our situation.

Thus it will suffice to check that $\mathcal{K}_T|_{\mathcal{U}} = \Phi^*(\mathcal{A}_0)|_{\mathcal{U}}$. The description of \mathcal{K}_T is very similar to that of \mathcal{A} . Let $\pi_L : \tilde{L} \rightarrow L$ be the Springer resolution, put $\tilde{\mathcal{U}} := \pi_L^{-1}(\mathcal{U})$, and we denote by $\pi_L^{\text{sr}} : \tilde{L}^{\text{sr}} \rightarrow L^{\text{sr}}$ the restriction of π to the preimage of L^{sr} . Then

the semisimple perverse sheaf \mathcal{K}_T is equal to $(\pi_L)_!(\overline{\mathbb{Q}}_l)[\dim L]$ over $\overline{\mathbb{F}}_q$, while the Weil structure of \mathcal{K}_T is induced (as in A.3) by isomorphism of functors $\mathrm{Fr}_w^*(\pi_L^{\mathrm{sr}})_! \xrightarrow{\sim} (\pi_L^{\mathrm{sr}})_!$.

By [BR, Thm. 6.2 and 9.1], there exists a L^{ad} -invariant open affine neighborhood $V \supset \mathcal{U}$ in L such that $\Phi|_V : V \rightarrow \mathcal{L}$ is étale. We claim that $\mathcal{K}_T|_V$ is isomorphic to $\Phi^*(\mathcal{A}_0)|_V = (\Phi|_V)^*(\mathcal{A}_0)$. As both $\mathcal{K}_T|_V$ and $(\Phi|_V)^*(\mathcal{A}_0)$ are semisimple perverse sheaves, which are immediate extensions of their restrictions to $V \cap \Phi^{-1}(\mathcal{L}^{\mathrm{sr}})$, it will suffice to show that $\Phi^*(\mathcal{A}_0)|_{\Phi^{-1}(\mathcal{L}^{\mathrm{sr}})} \cong \mathcal{K}_T|_{\Phi^{-1}(\mathcal{L}^{\mathrm{sr}})}$.

Note that $\Phi^{-1}(\mathcal{L}^{\mathrm{sr}}) \subset L^{\mathrm{sr}}$ and that Φ gives rise to the commutative diagram

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\tilde{\Phi}} & \tilde{\mathcal{L}} \\ \pi_L \downarrow & & \downarrow \pi \\ L & \xrightarrow{\Phi} & \mathcal{L} \end{array}$$

(use Lemma 1.8.3 (a)), whose restriction to $\mathcal{L}^{\mathrm{sr}}$ is Cartesian and W -equivariant. Therefore the required isomorphism $\Phi^*(\mathcal{A}_0)|_{\Phi^{-1}(\mathcal{L}^{\mathrm{sr}})} \xrightarrow{\sim} \mathcal{K}_T|_{\Phi^{-1}(\mathcal{L}^{\mathrm{sr}})}$ follows from the proper base change theorem.

□

APPENDIX B. $(a, a'; [b])$ -EQUIVALENCE AND FOURIER TRANSFORM

B.1. Formulation of the result.

B.1.1. Let G be a reductive group over a local field E of characteristic zero, $\mathcal{E} = (H, [\eta], \bar{s})$ an endoscopic triple for G , and $\varphi : G \rightarrow G'$ an inner twisting. Fix a triple $(a, a'; [b])$, where $a : T \hookrightarrow G$ and $a' : T \hookrightarrow G'$ are stably conjugate embeddings of maximal tori, and $[b]$ is a stable conjugacy class of embeddings of maximal tori $T \hookrightarrow H$, compatible with a and a' .

Fix a non-trivial character $\psi : E \rightarrow \mathbb{C}^\times$, a non-degenerate G -invariant pairing $\langle \cdot, \cdot \rangle$ on \mathcal{G} , and a non-zero translation invariant top degree differential form ω_G on G . Denote by $\langle \cdot, \cdot \rangle'$ the G' -invariant pairing on \mathcal{G}' , induced $\langle \cdot, \cdot \rangle$, and let $dx = |\omega_G|$ and $dx' = |\omega_{G'}|$ be the invariant measures on $\mathcal{G}(E)$ and $\mathcal{G}'(E)$ induced by ω_G . These data define Fourier transforms $F \mapsto \mathcal{F}(F)$ on $\mathcal{G}(E)$ and $F' \mapsto \mathcal{F}(F')$ on $\mathcal{G}'(E)$ (see 2.2.10).

The following result generalizes both the theorem of Waldspurger [Wa2] (who treated the case $\phi' = 0$) and of Kazhdan–Polishchuk [KP, Thm. 2.7.1] (where the stable case is treated).

Theorem B.1.2. *Generalized functions $F \in \mathcal{D}(\mathcal{G}(E))$ and $F' \in \mathcal{D}(\mathcal{G}'(E))$ are $(a, a'; [b])$ -equivalent if and only if $e'(G)\mathcal{F}(F)$ and $e'(G')\mathcal{F}(F')$ are $(a, a'; [b])$ -equivalent.*

Remark B.1.3. When this work was already written, we have learned that our Theorem B.1.2 seems to follow from the recent work of Chaudouard [Ch].

By duality, Theorem B.1.2 follows from the following result.

Theorem B.1.4. *Measures $\phi \in \mathcal{S}(\mathcal{G}(E))$ and $\phi' \in \mathcal{S}(\mathcal{G}'(E))$ are $(a, a'; [b])$ -indistinguishable if and only if $e'(G)\mathcal{F}(\phi)$ and $e'(G')\mathcal{F}(\phi')$ are $(a, a'; [b])$ -indistinguishable.*

For the proof, we will combine arguments from [Wa1, Wa2] with those from [KP].

Lemma B.1.5. (a) *The validity of Theorem B.1.4 is independent of the choice of ω_G , ψ and $\langle \cdot, \cdot \rangle$.*

(b) *It will suffice to show Theorem B.1.4 under the assumption that $G = G^{\text{sc}}$.*

(c) *It will suffice to show Theorem B.1.4 under the assumption that \mathcal{E} is elliptic. (Note that this is the only case used in this paper).*

Proof. The proof follows by essentially the same arguments as [Wa2, II].

(a) Another choice of the data results in replacing the Fourier transform \mathcal{F} by $B^* \circ \mathcal{F}$ for a certain linear automorphism B of \mathcal{G} commuting with $\text{Ad } G$. Thus the assertion follows from Lemma 1.6.15.

(b) Let $\tilde{\mathcal{E}}$ be the endoscopic triple for G^{sc} induced by \mathcal{E} , and let $(\tilde{a}, \tilde{a}'; [\tilde{b}])$ be the lift of $(a, a'; [b])$ (see Lemma 1.3.10). Fix a pair of $(a, a'; [b])$ -indistinguishable measures $\phi \in \mathcal{S}(\mathcal{G}(E))$ and $\phi' \in \mathcal{S}(\mathcal{G}'(E))$.

Denote by \mathcal{Z} the Lie algebra of $Z(G) = Z(G')$. Then $\mathcal{G} = \mathcal{G}^{\text{sc}} \oplus \mathcal{Z}$ and $\mathcal{G}' = \mathcal{G}'^{\text{sc}} \oplus \mathcal{Z}$, hence there exist measures $h_i \in \mathcal{S}(\mathcal{Z}(E))$, $f_i \in \mathcal{S}(\mathcal{G}^{\text{sc}}(E))$ and $f'_i \in \mathcal{S}(\mathcal{G}'^{\text{sc}}(E))$ such that the h_i 's are linearly independent, $\phi = \sum_i f_i \times h_i$ and $\phi' = \sum_i f'_i \times h_i$.

For each $x \in (\mathcal{G}^{\text{sc}})^{\text{sr}}(E)$, $z \in \mathcal{Z}(E)$ and $\bar{\kappa} \in \pi_0(\widehat{G_{x+z}}^\Gamma / Z(\widehat{G})^\Gamma)$, we have $O_{x+z}^{\bar{\kappa}}(\phi) = \sum_i O_x^{\bar{\kappa}}(f_i) O_z(h_i)$, where $\bar{\kappa} \in \pi_0(\widehat{G_x^{\text{sc}}}^\Gamma)$ is the image of $\bar{\kappa}$, and similarly for ϕ' . Since the h_i 's are linearly independent, it follows from Lemmas 1.6.11 and 1.3.10 that f_i and f'_i are $(\tilde{a}, \tilde{a}'; [\tilde{b}])$ -indistinguishable for each i (compare Corollary 1.6.13 and its proof).

Let $\langle \cdot, \cdot \rangle$ be a direct sum of pairings on \mathcal{G}^{sc} and \mathcal{Z} . Then $\mathcal{F}(\phi) = \sum_i \mathcal{F}(f_i) \times \mathcal{F}(h_i)$ and $\mathcal{F}(\phi') = \sum_i \mathcal{F}(f'_i) \times \mathcal{F}(h_i)$. By our assumptions, $e'(G)\mathcal{F}(f_i)$ and $e'(G')\mathcal{F}(f'_i)$ are $(\tilde{a}, \tilde{a}'; [\tilde{b}])$ -indistinguishable for each i , therefore $e'(G)\mathcal{F}(\phi)$ and $e'(G')\mathcal{F}(\phi')$ are $(a, a'; [b])$ -indistinguishable, as claimed.

(c) The assertion follows from the arguments of [Wa2, II. 3]. Since we do not use this result in the main body of the paper, we omit the details. \square

From now on, we assume that \mathcal{E} is elliptic, G is semisimple and simply connected, and $\langle \cdot, \cdot \rangle$ is the Killing form.

Notation B.1.6. (a) Consider the natural map $[y] \mapsto [y]_G$ from the set of stable conjugacy classes of elements of $\mathcal{H}^{\text{sr}}(E)$ to the set of E -rational conjugacy classes

in $\mathcal{G}^{\text{sr}}(\overline{E})$, defined as follows. For each $y \in \mathcal{H}^{\text{sr}}(E)$, denote by b_y the inclusion $H_y \hookrightarrow H$. Each embedding $a : (H_y)_{\overline{E}} \hookrightarrow G_{\overline{E}}$ from $[b_y]_G$ defines an embedding $da : (\mathcal{H}_y)_{\overline{E}} \hookrightarrow \mathcal{G}_{\overline{E}}$, and we denote by $[y]_G$ be the conjugacy class of $da(y) \in \mathcal{G}(\overline{E})$.

(b) We say that $y \in \mathcal{H}^{\text{sr}}(E)$ and $x \in \mathcal{G}^{\text{sr}}(E)$ are *compatible* if $x \in [y]_G$.

(c) For each $x \in \mathcal{G}^{\text{sr}}(E)$, the map $b \mapsto y := db(x)$ defines a bijection between embeddings of maximal tori $b : G_x \hookrightarrow H$ compatible with $a_x : G_x \hookrightarrow G$ and elements $y \in \mathcal{H}^{\text{sr}}(E)$ compatible with x . Let $y \mapsto b_y$ be the inverse map.

We will write $\overline{\kappa}_{[y]} \in \pi_0(\widehat{G}_x^\Gamma)$ instead of $\overline{\kappa}_{[b_y]}$, $O_x^{[y]}$ instead of $O_x^{\overline{\kappa}_{[y]}}$, and $\left\langle \frac{x, x'; [y]}{a, a'; [b]} \right\rangle$, where $x' \in \mathcal{G}'(E)$ is a stable conjugate of x , instead of $\left\langle \frac{a_x, a_{x'}; [b_y]}{a, a'; [b]} \right\rangle$.

B.2. Local calculations.

The primary goal of this subsection is to construct $(a, a'; [b])$ -indistinguishable measures whose Fourier transforms are $(a, a'; [b])$ -indistinguishable in some region. We mostly follow [KP].

B.2.1. (a) For each $t \in \mathcal{G}^{\text{sr}}(E)$, fix a top degree form $\omega_{\mathcal{G}_t} \neq 0$ on the vector space \mathcal{G}_t and identify it with the corresponding top degree translation invariant differential form. Then $\omega_{\mathcal{G}_t}$ defines a G_t -invariant top degree form $\nu = \nu_t := \omega_{\mathcal{G}} \otimes (\omega_{\mathcal{G}_t})^{-1}$ on $\mathcal{G}/\mathcal{G}_t$, which uniquely extends to a non-zero top degree G -invariant form on G/G_t , which we will also denote by ν .

(b) We denote by $d\overline{g}$ and du the measures $|\nu|$ on $(G/G_t)(E)$ and $|\omega_{\mathcal{G}_t}|$ on $\mathcal{G}_t(E)$, respectively.

(c) Consider the map $\Pi : (G/G_t) \times \mathcal{G}_t^{\text{sr}} \rightarrow \mathcal{G}^{\text{sr}}$ given by the rule $\Pi(\overline{g}, x) = \text{Ad } \overline{g}(x)$. Then Π is étale, and we have an equality $\Pi^*(\omega_{\mathcal{G}}) = \nu \wedge \omega_{\mathcal{G}_t}$.

B.2.2. Preliminaries on quadratic forms over local fields. (a) To every non-degenerate quadratic form q on an E -vector space V one associates a rank $\text{rk } q = \dim V$, a determinant $\det q \in (\det V)^{\otimes(-2)}$ and a Hasse–Witt invariant $e(q) \in \{-1, 1\}$. Any trivialization $\det V \xrightarrow{\sim} E$ associates to $\det q$ an element of E^\times . Moreover, the class of $\det q$ in $E^\times/(E^\times)^2$ is independent on the trivialization.

To each isomorphism class (q, ψ) , where ψ is a non-trivial additive character of E , Weil [We] associated an 8th root of unity $\gamma(q, \psi)$. For each non-zero top degree form ν on V , we set $c(q, \nu, \psi) := \gamma(q, \psi) |\det(q)/\nu^2|^{-1/2}$.

(b) Weil proved that for every non-degenerate quadratic forms q and q' satisfying $\text{rk } q = \text{rk } q'$ and $\det q \equiv \det q' \pmod{(E^\times)^2}$, we have $\gamma(q, \psi)/\gamma(q', \psi) = e(q)e(q')$.

(c) To each $t \in \mathcal{G}^{\text{sr}}(E)$ and $y, z \in \mathcal{G}_t^{\text{sr}}(E)$ we associate a non-degenerate quadratic form $q = q_{y,z} : \overline{x} \mapsto \langle y, (\text{ad } \overline{x})^2(z) \rangle$ on $V := \mathcal{G}/\mathcal{G}_t$. Then the form ν on V , chosen in B.2.1 (a) gives rise to an invariant $c(q_{y,z}, \nu, \psi)$.

(d) Let $t' \in \mathcal{G}'^{\text{sr}}(E)$ be a stable conjugate of t , ν' the form on $G'/G'_{t'}$ induced by ν , and $\varphi_{t,t'}$ the canonical isomorphism $\mathcal{G}_t \xrightarrow{\sim} \mathcal{G}_{t'}$ from Construction 1.7.7 (b). For each $y, z \in \mathcal{G}_t^{\text{sr}}$, we set $y' = \varphi_{t,t'}(y)$ and $z' = \varphi_{t,t'}(z)$.

It follows from results of [KP] that $c(q_{y,z}, \nu, \psi)/c(q_{y',z'}, \nu', \psi) = e'(G)e'(G')$. In particular, $c(q_{y,z}, \nu, \psi) = c(q_{y',z'}, \nu', \psi)$ if $G' = G$. Indeed, since $\det q_{y,z}/\nu^2 = \det q_{y',z'}/\nu'^2$ and $\text{rk } q = \text{rk } q'$, the assertion is a combination of the result of Weil (see (b)) and [KP, Lem. 2.7.5 and the remark following it].

Notation B.2.3. Fix $t \in \mathcal{G}^{\text{sr}}(E)$ and sufficiently small open compact subgroups $K \subset G(E)$ and $U \subset \mathcal{G}_t(E)$ such that $t + U \subset \mathcal{G}_t^{\text{sr}}(E)$ and $K \cap \text{Norm}_{G(E)}(\mathcal{G}_t) = K \cap G_t(E)$.

For each $a \in E^\times$, put $S_a := \text{Ad } K(at + U) \subset \mathcal{G}(E)$, and we denote by $\chi_a = \chi_{a,t,K,U}$ be the characteristic function of S_a .

B.2.4. Stationary phase principle. For each $u \in \mathcal{G}^{\text{sr}}(E)$ and $x \in \mathcal{G}(E)$, we define function $f_{x,u} : (G/G_u)(E) \rightarrow E$ by the rule $f_{x,u}(\bar{g}) := \langle x, \text{Ad } \bar{g}(u) \rangle$. Then $\bar{g} \in (G/G_u)(E)$ is a critical point for $f_{x,u}$ if and only if $x \in \text{Ad } \bar{g}(\mathcal{G}_u) = \mathcal{G}_{\text{Ad } \bar{g}(u)}$. In this case, the corresponding quadratic form on $T_{\bar{g}}(G/G_u) = \mathcal{G}/\mathcal{G}_u$ is $q_{\text{Ad } \bar{g}^{-1}(x),u}$.

By the stationary phase principle (see [KP, Lem 2.5.1]), for each compact subset $C \subset G^{\text{sr}}(E)$ there exists $N_0 = N_0(t, K, U, c) \in \mathbb{N}$ such that for each $x \in C$, $u \in t + U$ and $a \in E^\times$ with $|a| > N_0$, the integral $\int_{K/K \cap G_t(E)} \psi(a \langle x, \text{Ad } \bar{g}(u) \rangle) d\bar{g}$ equals

$$(B.2.1) \quad c(q_{\text{Ad } \bar{k}(x),u}, \nu, \psi) \psi(a \langle x, u \rangle) |a|^{-\frac{1}{2} \dim \mathcal{G}/\mathcal{G}_t},$$

if there exists (a unique) element $\bar{k} \in K/K \cap G_t(E)$ such that $\text{Ad } \bar{k}(x) \in \mathcal{G}_t^{\text{sr}}(E)$, and vanishes otherwise.

Indeed, the map $f_{x,u}|_{K/K \cap G_t(E)}$ has a unique non-degenerate critical point $\bar{g} = \bar{k}^{-1}$ in the former case and has no critical points in the latter one. Thus the assertion for $a \in (E^\times)^2$ follows from [KP, Lem 2.5.1]. Since the quotient $E^\times/(E^\times)^2$ is finite, the general case now follows from the previous one applied to the compact set $\sqcup_b bC$, where $b \in E^\times$ runs over a set of representatives of $E^\times/(E^\times)^2$.

Lemma B.2.5. Let t, K and U be as in Notation B.2.3, and let $[x] \subset \mathcal{G}^{\text{sr}}(E)$ be a stable conjugacy class. Then there exists $N = N(t, K, U, [x]) \in \mathbb{N}$ such that for each $x \in [x]$ and $a \in E^\times$ with $|a| > N$, the Fourier transform $\mathcal{F}(\chi_a)(x)$ equals

$$(B.2.2) \quad c(q_{\text{Ad } \bar{k}(x),t}, \nu, \psi) \psi(a \langle x, t \rangle) |a|^{\frac{1}{2} \dim \mathcal{G}/\mathcal{G}_t} \int_U \psi(\langle x, u \rangle) du,$$

if there exists (a unique) element $\bar{k} \in K/K \cap G_t(E)$ such that $\text{Ad } \bar{k}(x) \in \mathcal{G}_t^{\text{sr}}(E)$, and vanishes otherwise.

Proof. First we claim that there exists a compact set $C_0 \subset \mathcal{G}(E)$ containing the support of $\mathcal{F}(\chi_a)$ for all $a \in E \setminus \mathcal{O}$. To show this we will find an \mathcal{O} -lattice $L \subset \mathcal{G}$ such that $S_a + L = S_a$ for all $a \in E \setminus \mathcal{O}$. Then the dual lattice $C_0 := \{x \in \mathcal{G} \mid \psi(\langle x, L \rangle) = 1\}$ satisfies the required property.

Recall that the map $(\bar{k}, u) \mapsto \text{Ad } k(t + u)$ gives an analytic isomorphism $F : (K/K \cap G_t(E)) \times U \xrightarrow{\sim} S_1$, and we denote by $\pi : S_1 \rightarrow U$ the composition $\text{pr}_2 \circ F^{-1}$. Since S_1 is compact, π has a bounded derivative. Therefore there exists a lattice $L \subset \mathcal{G}(E)$ such that for each $b \in E$ and $x \in S_1$, we have $\pi(x + bL) \subset \pi(x) + bU$. Shrinking L if necessary, we can moreover assume that $S_1 + L = S_1$.

Fix $a \in E \setminus \mathcal{O}$. Then $a^{-1}S_a = \text{Ad } K(t + a^{-1}U) \subset S_1$ and, moreover, $a^{-1}S_a$ is the preimage of $a^{-1}U \subset U$ under π . Therefore $a^{-1}S_a + a^{-1}L \subset S_1 + L = S_1$ and $\pi(a^{-1}S_a + a^{-1}L) \subset \pi(a^{-1}S_a) + a^{-1}U = a^{-1}U$. Hence $a^{-1}S_a + a^{-1}L \subset \pi^{-1}(a^{-1}U) = a^{-1}S_a$, thus $S_a + L = S_a$, as claimed.

As the intersection $\mathcal{G}_t^{\text{sr}} \cap [x]$ is finite, the set $\text{Ad } K(\mathcal{G}_t^{\text{sr}} \cap [x])$ and therefore also $C := C_0 \cup \text{Ad } K(\mathcal{G}_t^{\text{sr}} \cap [x])$ is compact. Since $[x] \subset \mathcal{G}(E)$ is closed, the intersection $C \cap [x]$ is compact as well. Take any $N \geq N_0(t, K, U, C \cap [x])$ (see B.2.4) such that quadratic forms $q_{x,u}$ and $q_{x,t}$ are isomorphic for each element x of the finite set $\mathcal{G}_t(E) \cap [x]$ and each $u \in t + a^{-1}U$ with $|a| > N$. We claim that this N satisfies the required properties.

Indeed, the Fourier transform $\mathcal{F}(\chi_a)(x) = \int_{S_a} \psi(\langle x, y \rangle) dy$ equals

$$|a|^{\dim \mathcal{G}} \int_{a^{-1}S_a} \psi(a \langle x, y \rangle) dy = |a|^{\dim \mathcal{G}} \int_{t+a^{-1}U} du \int_{K/K \cap G_t(E)} \psi(x \langle a, \text{Ad } \bar{g}(u) \rangle) d\bar{g}$$

(by B.2.1 (c) and our assumptions in Notation B.2.3). Therefore the assertion for $x \in [x] \cap C$ follows from B.2.4. Finally, if $x \in [x] \setminus C$, then $x \notin \text{Ad } K(\mathcal{G}_t^{\text{sr}})$ and $\mathcal{F}(\chi_a)(x) = 0$, implying the assertion in the remaining case. \square

Corollary B.2.6. *For each triple (y, x, x') , where $x \in \mathcal{G}^{\text{sr}}(E)$ and $x' \in \mathcal{G}'^{\text{sr}}(E)$ are stably conjugate, and $y \in \mathcal{H}^{\text{sr}}(E)$ is compatible with x , there exist measures $\phi \in \mathcal{S}(\mathcal{G}(E))$ and $\phi' \in \mathcal{S}(\mathcal{G}'(E))$ satisfying the following properties:*

- (i) ϕ and ϕ' are supported on elements stably conjugate to $\mathcal{G}_x^{\text{sr}}(E)$.
- (ii) $O_u^{[y]}(\phi) = \left\langle \frac{x, x'; [y]}{a, a'; [b]} \right\rangle O_{u'}^{[y]}(\phi')$ for each $u \in \mathcal{G}_x^{\text{sr}}(E)$ and $u' = \phi_{x, x'}(u) \in \mathcal{G}_{x'}'^{\text{sr}}(E)$.
- (iii) $O_u^{\bar{\xi}}(\phi) = O_{u'}^{\bar{\xi}}(\phi') = 0$ for each $u \in \mathcal{G}_x^{\text{sr}}(E)$, $u' = \phi_{x, x'}(u)$ and $\bar{\xi} \neq \bar{\kappa}_{[y]}$.
- (iv) $O_x^{\bar{\xi}}(\mathcal{F}(\phi)) = O_{x'}^{\bar{\xi}}(\mathcal{F}(\phi')) = 0$ for each $\bar{\xi} \neq \bar{\kappa}_{[y]}$.
- (v) $e'(G)O_x^{[y]}(\mathcal{F}(\phi)) = e'(G') \left\langle \frac{x, x'; [y]}{a, a'; [b]} \right\rangle O_{x'}^{[y]}(\mathcal{F}(\phi')) \neq 0$.

Proof. (compare Corollary 1.7.8). Let $x_1 = x, x_2, \dots, x_l$ be all elements of $\mathcal{G}_x(E) \cap [x]$. Pick $t \in \mathcal{G}_x^{\text{sr}}(E)$ such that all $\langle x_j, t \rangle$ are distinct, and put $t' := \phi_{x, x'}(t) \in \mathcal{G}_{x'}'^{\text{sr}}(E)$. Choose sufficiently small subgroups $K \subset G(E)$, $K' \subset G'(E)$ and $U \subset \mathcal{G}_t(E)$

satisfying the assumptions of Notation B.2.3 and such that $\psi(\langle x_j, u \rangle) = 1$ for each $u \in U$ and each $j = 1, \dots, l$.

Let $t_1 = t, t_2, \dots, t_k \in \mathcal{G}(E)$ and $t'_1 = t', t'_2, \dots, t'_k \in \mathcal{G}'(E)$ be sets of representatives of conjugacy classes stably conjugate to t and t' . For each $i = 1, \dots, k$, put $U_i := \varphi_{t, t_i}(U) \subset \mathcal{G}_{t_i}(E)$ and $U'_i := \varphi_{t, t'_i}(U) \subset \mathcal{G}'_{t'_i}(E)$.

For each $i = 1, \dots, k$, let ν_i and ν'_i be the forms on G/G_{t_i} and $G'/G'_{t'_i}$ respectively, induced by ν . For each $a \in E \setminus \mathcal{O}$, put $\phi_i = \frac{1}{|\nu_i|(K/K \cap G_{t_i}(E))} \chi_{a, t_i, U_i, K} dx$ and $\phi'_i = \frac{1}{|\nu'_i|(K'/K' \cap G'_{t'_i}(E))} \chi_{a, t'_i, U'_i, K'} dx'$. Define ϕ and ϕ' by the formulas $\phi :=$

$\sum_i \langle \text{inv}(t, t_i), \bar{\kappa}_{[y]} \rangle^{-1} \phi_i$ and $\phi' := \sum_i \left\langle \frac{a t, a t'_i; [b_y]}{a, a'; [b]} \right\rangle^{-1} \phi'_i$. Then ϕ and ϕ' clearly satisfy properties (i)–(iii) for each $a \in E \setminus \mathcal{O}$, so it remains to show the existence of a for which properties (iv) and (v) are satisfied.

Let N be the maximum of the $N(t_i, U_i, K, [x])$'s and the $N(t'_i, U'_i, K', [x'])$'s. Then Lemma B.2.5 implies that for each $\bar{\xi} \in \pi_0(\widehat{G_x}^\Gamma)$ and $a \in E^\times$ with $|a| > N$, we have

$$(B.2.3) \quad O_x^{\bar{\xi}}(\mathcal{F}(\phi_1)) = |a|^{\frac{1}{2} \dim \mathcal{G}/\mathcal{G}_t} |\omega_{\mathcal{G}_t}|(U) \sum_{j=1}^l \langle \text{inv}(x, x_j), \bar{\xi} \rangle c(q_{x_j, t}, \nu, \psi) \psi(a \langle x_j, t \rangle).$$

Moreover, for each $i = 1, \dots, n$, observation B.2.2 (d) and the equality $\varphi_{t, t_i}^*(\omega_{\mathcal{G}_{t_i}}) = \omega_{\mathcal{G}_t}$ imply that

$$(B.2.4) \quad O_x^{\bar{\xi}}(\mathcal{F}(\phi_i)) = \langle \text{inv}(t, t_i), \bar{\xi} \rangle O_x^{\bar{\xi}}(\mathcal{F}(\phi_1)).$$

Then (B.2.3) and (B.2.4) together with similar formulas for the ϕ'_i 's imply the equalities from (iv) and (v) and that $O_x^{[y]}(\mathcal{F}(\phi)) = k O_x^{[y]}(\mathcal{F}(\phi_1))$.

It remains to show the existence of $a \in E^\times$ with $|a| > N$ such that $O_x^{[y]}(\mathcal{F}(\phi_1)) \neq 0$. Since all $\langle x_j, t \rangle$ are distinct, the functions $a \mapsto \psi(a \langle x_j, t \rangle)$ are linearly independent. The assertion now follows from (B.2.3). \square

Later we will need the following result

Lemma B.2.7. *Let G be an unramified semisimple simply connected group over E , $K \subset G(E)$ a hyperspecial subgroup, $\mathcal{K} \subset \mathcal{G}(E)$ the corresponding subalgebra, $1_{\mathcal{K}}$ the characteristic function of \mathcal{K} , and $\mathcal{E} = (H, [\eta], \bar{s})$ an endoscopic triple for G .*

(a) *If H is ramified, then $O_x^{[y]}(1_{\mathcal{K}} dx) = 0$ for all compatible elements $y \in \mathcal{H}^{\text{sr}}(E)$ and $x \in \mathcal{G}^{\text{sr}}(E)$.*

(b) *If H is unramified, then there exists an open neighborhood of zero $\Omega \subset \mathcal{H}(E)$ such that $O_x^{[y]}(1_{\mathcal{K}} dx) \neq 0$ for all compatible elements $y \in \Omega \cap \mathcal{H}^{\text{sr}}(E)$ and $x \in \mathcal{G}^{\text{sr}}(E)$.*

(c) If $x \in \mathcal{K}$ has a regular reduction modulo \mathfrak{m} , then every stably conjugate $x' \in \mathcal{K}$ of x is K -conjugate. In particular, $O_x^{\bar{\xi}}(1_K dx) \neq 0$ for each $\bar{\xi} \in \pi_0(\widehat{G_x}^\Gamma)$.

Proof. Since in the notation of [Wa1, Wa2], $O_x^{[y]}(1_K dx)$ is a non-zero multiple of $J^{G,H}(y, 1_K)$, the assertions follow from [Wa1, 7.2 and 7.4] and [Wa2, III, Prop.] (compare [Ko2, Prop. 7.1 and 7.5]). \square

B.3. Global results.

For the proof of Theorem B.1.4 we will use global methods. In this subsection we will recall necessary notation and results.

B.3.1. (a) Let \underline{E} be a number field, which we will always assume to be totally imaginary, Γ the absolute Galois group of \underline{E} , and \mathbb{A} the ring of adèles of \underline{E} . We denote by V, V_∞ and V_f the set of all places, all infinite places and all finite places of \underline{E} , respectively. For each $v \in V_f$, we have a natural conjugacy class of embeddings $\Gamma_v \hookrightarrow \Gamma$. For every object \underline{S} over \underline{E} and each $v \in V$, we will denote by \underline{S}_v the corresponding object over \underline{E}_v .

(b) For every reductive group \underline{G} over \underline{E} , consider a sequence

$$(B.3.1) \quad H^1(\underline{E}, \underline{G}) \rightarrow \bigoplus_{v \in V_f} H^1(\underline{E}_v, \underline{G}) \xrightarrow{*} \pi_0(Z(\widehat{\underline{G}})^\Gamma)^D,$$

where the restriction of $*$ to $H^1(\underline{E}_v, \underline{G})$ is the composition of the isomorphism $\mathcal{D}_G : H^1(\underline{E}_v, \underline{G}) \xrightarrow{\sim} \pi_0(Z(\widehat{\underline{G}})^{\Gamma_v})^D$ from 1.5.1 and the projection $\pi_0(Z(\widehat{\underline{G}})^{\Gamma_v})^D \rightarrow \pi_0(Z(\widehat{\underline{G}})^\Gamma)^D$. Kottwitz proved (see [Ko2, Prop 2.6]) that the sequence (B.3.1) is exact.

(c) For each $c \in H^1(\underline{E}, \underline{G})$, $\kappa \in \pi_0(Z(\widehat{\underline{G}})^\Gamma)$ and $v \in V_f$, we denote by $c_v \in H^1(\underline{E}_v, \underline{G})$ and $\kappa_v \in \pi_0(Z(\widehat{\underline{G}})^{\Gamma_v})$ the images of c and κ , respectively. Then Kottwitz' theorem from (b) asserts that $\prod_{v \in V_f} \langle c_v, \kappa_v \rangle = 1$.

Lemma B.3.2. Let \underline{G} be a reductive group over \underline{E} , and $u \in V_f$.

(a) If u is inert in the splitting field $\underline{E}[\underline{G}^*]$ of the quasi-split inner form \underline{G}^* of \underline{G} , then the diagonal map $H^1(\underline{E}, \underline{G}) \rightarrow \bigoplus_{v \neq u} H^1(\underline{E}_v, \underline{G})$ is surjective.

Assume in addition that \underline{G} is either semisimple and simply connected or adjoint. Then

(b) The map from (a) is an isomorphism.

(c) Let \underline{T} be a maximal torus of \underline{G} , and let $c \in H^1(\underline{E}, \underline{G})$ be such that $c_v \in H^1(\underline{E}_v, \underline{G})$ belongs to $\text{Im}[H^1(\underline{E}_v, \underline{T}) \rightarrow H^1(\underline{E}_v, \underline{G})]$ for each $v \neq u$. Then $c \in \text{Im}[H^1(\underline{E}, \underline{T}) \rightarrow H^1(\underline{E}, \underline{G})]$ in each of the following cases:

- (i) u is inert in the splitting field $\underline{E}[\underline{T}]$ of \underline{T} ;
- (ii) u inert in $\underline{E}[\underline{G}^*]$, and $\underline{T}_u \subset \underline{G}_u$ is elliptic.

Proof. (a), (b) By assumption, we have $Z(\widehat{G})^\Gamma = Z(\widehat{G})^{\Gamma_u}$, so assertion (a) follows from the exactness of (B.3.1) while assertion (b) follows from the Hasse principle.

(c) Consider commutative diagram

$$\begin{array}{ccc} H^1(\underline{E}, \underline{T}) & \xrightarrow{A} & \bigoplus_{v \neq u} H^1(\underline{E}_v, \underline{T}) \\ \downarrow & & \downarrow D \\ H^1(\underline{E}, \underline{G}) & \xrightarrow{B} & \bigoplus_{v \neq u} H^1(\underline{E}_v, \underline{G}). \end{array}$$

In both cases, u is inert in $\underline{E}[\underline{G}^*]$, hence B is injective (by (b)). Since by our assumption, $B(c) = (c_v)_{v \neq u}$ belongs to $\text{Im } D$, it will suffice to show that A is surjective. In the case (i), the surjectivity of A follows from (a). In the case (ii), the canonical map $\pi_0(\widehat{T}^\Gamma) = \widehat{T}^\Gamma / Z(\widehat{G})^\Gamma \hookrightarrow \widehat{T}^{\Gamma_v} / Z(\widehat{G})^{\Gamma_v} = \pi_0(\widehat{T}^{\Gamma_u})$ is injective. Therefore surjectivity of A follows from the exactness of (B.3.1). \square

From now on, \underline{G} is a semisimple and simply connected group over \underline{E} .

B.3.3. (a) Let $\mathcal{E}' = (H', [\eta'], \overline{s}')$ be an endoscopic triple for \underline{G} . For each $v \in V_f$, \mathcal{E}' gives rise to an (isomorphism class of an) endoscopic triple $\mathcal{E}'_v = (H'_v, [\eta'_v], \overline{s}'_v)$ for \underline{G}_v .

In particular, if $\mathcal{E}' \cong \mathcal{E}_{([\underline{a}], \underline{\kappa})}$ for a certain pair $(\underline{a}, \underline{\kappa})$ consisting of an embedding of a maximal torus $\underline{a} : \underline{T} \hookrightarrow \underline{G}$ and $\underline{\kappa} \in \widehat{T}^\Gamma$, then $\mathcal{E}'_v \cong \mathcal{E}_{([\underline{a}_v], \underline{\kappa}_v)}$.

(b) Let $\mathcal{E}'_i = (H'_i, [\eta'_i], \overline{s}'_i)$, $i = 1, 2$ be a pair of endoscopic triples for \underline{G} . Assume that there exists $v \in V_f$, inert in both splitting fields $\underline{E}[H'_1]$ and $\underline{E}[H'_2]$, such that $(\mathcal{E}'_1)_v \cong (\mathcal{E}'_2)_v$. Then $\mathcal{E}'_1 \cong \mathcal{E}'_2$.

Indeed, the image of $\rho_{H'_i} : \Gamma \rightarrow \text{Out}(\widehat{H'_i})$ coincides with that of $\rho_{(H'_i)_v} : \Gamma_v \rightarrow \text{Out}(\widehat{H'_i})$ for each $i = 1, 2$, therefore by Remark 1.3.4, the map

$$\text{Isom}(\mathcal{E}'_1, \mathcal{E}'_2) / (H'_1)^{\text{ad}}(\underline{E}) \rightarrow \text{Isom}((\mathcal{E}'_1)_v, (\mathcal{E}'_2)_v) / (H'_1)^{\text{ad}}(\underline{E}_v)$$

is bijective.

(c) Let $\mathcal{E}' = (H', [\eta'], \overline{s}')$ be an endoscopic triple for G , $\varphi : \underline{G} \rightarrow \underline{G}'$ an inner twisting, and $(\underline{a}'_i, \underline{a}_i, [\underline{b}_i])$ be two triples consisting of stably conjugate embeddings of maximal tori $\underline{a}_i : \underline{T}_i \hookrightarrow \underline{G}$ and $\underline{a}'_i : \underline{T}_i \hookrightarrow \underline{G}'$, and stably conjugate classes $[\underline{b}_i]$ of embeddings of maximal tori $\underline{T}_i \hookrightarrow H'$, compatible with \underline{a}_i . Then we have the following product formula

$$\prod_{v \in V_f} \left\langle \frac{(\underline{a}_1)_v, (\underline{a}'_1)_v; [\underline{b}_1]_v}{(\underline{a}_2)_v, (\underline{a}'_2)_v; [\underline{b}_2]_v} \right\rangle_{\mathcal{E}'_v} = 1.$$

Indeed, consider elements $c := \overline{\text{inv}}((\underline{a}_1, \underline{a}'_1); (\underline{a}_2, \underline{a}'_2)) \in H^1(\underline{E}, (\underline{T}_1 \times \underline{T}_2) / Z(\underline{G}))$ (see 1.1.5) and $\kappa := \kappa([\underline{b}_1], [\underline{b}_2])(\overline{s}') \in \pi_0(\{(\underline{T}_1 \times \underline{T}_2) / Z(\underline{G})\}^\wedge)^\Gamma$ (see 1.2.8). Since

$\left\langle \frac{(\underline{a}_1)_v, (\underline{a}'_1)_v; [\underline{b}_1]_v}{(\underline{a}_2)_v, (\underline{a}'_2)_v; [\underline{b}_2]_v} \right\rangle_{\mathcal{E}'_v} = \langle c_v, \kappa_v \rangle$ for each $v \in V_f$, the product formula follows from Kottwitz' theorem (see B.3.1 (c)).

B.3.4. Denote by dg and $dx = \prod_v dx_v$ the Tamagawa measures on $\underline{G}(\mathbb{A})$ and $\underline{\mathcal{G}}(\mathbb{A})$, respectively (defined by a non-zero translation invariant top degree differential form $\omega_{\underline{G}}$ on \underline{G}).

For each $v \in V_\infty$, denote by $\mathcal{S}(\underline{G}(\underline{E}_v))$ the space of measures on $\underline{G}(\underline{E}_v)$ of the form $f_v dx_v$, where f_v is a Schwartz function. Put $\mathcal{S}(\underline{\mathcal{G}}(\mathbb{A})) := \otimes'_{v \in V} \mathcal{S}(\underline{G}(\underline{E}_v))$, and fix a non-trivial character $\underline{\psi} = \prod_v \psi_v : \mathbb{A}/\underline{E} \rightarrow \mathbb{C}^\times$. Then $\underline{\psi}$ gives rise to a Fourier transform $\mathcal{F} : \mathcal{S}(\underline{\mathcal{G}}(\mathbb{A})) \rightarrow \underline{\mathcal{S}}(\underline{\mathcal{G}}(\mathbb{A}))$ such that $\mathcal{F}(\otimes_v \phi_v) = \otimes_v \mathcal{F}(\phi_v)$, where $\mathcal{F}(\phi_v)$ is the Fourier transform of ϕ_v corresponding to a measure dx_v and a character ψ_v .

For each $\underline{x} \in \underline{\mathcal{G}}^{\text{sr}}(\underline{E})$, $\underline{\kappa} \in \widehat{\underline{G}}_{\underline{x}}^\Gamma$, and $\phi = \otimes_v \phi_v \in \mathcal{S}(\underline{\mathcal{G}}(\mathbb{A}))$ put $O_{\underline{x}}^{\underline{\kappa}}(\phi) := \prod_{v \in V} O_{\underline{x}_v}^{\underline{\kappa}_v}(\phi_v)$, where $O_{\underline{x}_v}^{\underline{\kappa}_v} := O_{\underline{x}_v}$ for each $x \in V_\infty$. It follows from Kottwitz' theorem (see B.3.1 (c)) that generalized function $O_{\underline{x}}^{\underline{\kappa}}$ depends only on the stable conjugacy class of \underline{x} .

The main technical tool for the proof of Theorem B.1.4 is the following simple version of the trace formula. Let θ be a generalized function on $\underline{\mathcal{G}}(\mathbb{A})$ defined by the rule $\theta(fdx) = \sum_{x \in \mathcal{G}(F)} f(x)$. For each $g \in \underline{G}(\mathbb{A})$, put $\theta^g := (\text{Ad } g)^*(\theta)$.

Proposition B.3.5. (a) Let $\phi = \otimes_v \phi_v$ be an element of $\mathcal{S}(\underline{\mathcal{G}}(\mathbb{A}))$ such that $\text{Supp}(\phi) \cap \text{Ad } \underline{G}(\mathbb{A})(\underline{\mathcal{G}}(\underline{E}))$ consists of regular elliptic elements. Then the integral

$$\Theta(\phi) := \int_{\underline{G}(\mathbb{A})/\underline{G}(\underline{E})} \theta^g(\phi) dg$$

converges absolutely. Furthermore,

$$\Theta(\phi) = \sum_{\underline{x}} \sum_{\underline{\kappa} \in \widehat{\underline{G}}_{\underline{x}}^\Gamma} O_{\underline{x}}^{\underline{\kappa}}(\phi),$$

where \underline{x} runs over the set of regular elliptic stably conjugacy classes of $\underline{\mathcal{G}}(\underline{E})$.

(b) If $\mathcal{F}(\phi)$ also satisfies the support assumption of (a), then $\Theta(\phi) = \Theta(\mathcal{F}(\phi))$.

Proof. (a) The first assertion (see [Wa1, 10.8]) is a direct analog of the corresponding result of Arthur, while the second one (see, for example, [KP, Thm 3.2.1]) is an analog of a result of Kottwitz.

(b) By the Poisson summation formula, we have $\mathcal{F}(\theta) = \theta$. Therefore the assertion follows from the absolute convergence of $\Theta(\phi)$ and $\Theta(\mathcal{F}(\phi))$. \square

To apply the trace formula, we will embed our local data into global ones.

Claim B.3.6. *There exist a totally imaginary number field \underline{E} , two finite places w and u of \underline{E} , a semisimple simply connected group \underline{G} over \underline{E} , an inner twisting $\varphi : \underline{G} \rightarrow \underline{G}'$, an endoscopic triple $\underline{\mathcal{E}} = (\underline{H}, [\eta], \underline{\mathfrak{s}})$ for \underline{G} , a tori \underline{T} over \underline{E} , a pair of stably conjugate embeddings of maximal tori $\underline{a} : \underline{T} \hookrightarrow \underline{H}$ and $\underline{a}' : \underline{T} \hookrightarrow \underline{G}'$, and an embedding $\underline{b} : \underline{T} \hookrightarrow \underline{G}$ compatible with \underline{a} satisfying the following conditions:*

(a) $\underline{E}_w \cong E$, $\underline{G}_w \cong G$, $\varphi_w \cong \varphi$, $\underline{\mathcal{E}}_w \cong \mathcal{E}$, $\underline{T}_w \cong T$, $[\underline{b}]_w = [b]$, while \underline{a}_w and \underline{a}'_w are conjugate to a and a' , respectively.

(b) For each $v \neq u, w$, the groups \underline{G}_v and \underline{G}'_v are quasi-split, and φ_v is trivial. Moreover, after we identify \underline{G}_v with \underline{G}'_v by means of some $\underline{G}(\overline{\underline{E}}_v)$ -conjugate of φ_v , embeddings \underline{a}_v and \underline{a}'_v are conjugate.

(c) u is inert in $\underline{E}[\underline{T}]$, and $\underline{\mathcal{E}}_u$ is elliptic.

Proof. (I) Put $E' := E[T]$, and set $\Gamma' := \text{Gal}(E'/E)$. Choose a dense subfield F' of E' , which is a finite extension of \mathbb{Q} . Increasing F' , we may assume that F' is Γ' -invariant. Set $F := (F')^{\Gamma'}$, and let w_0 be the prime of F , corresponding to the embedding $F \hookrightarrow (E')^{\Gamma'} = E$. In particular, $F_{w_0} \cong E$. Choose a totally imaginary quadratic extension \underline{E} of F_0 such that w_0 splits in \underline{E} , and let w and u be the primes of \underline{E} lying over w_0 . Finally, let \underline{E}' be the composite field $\underline{E} \cdot F'$. We have natural identifications $\underline{E}_w \cong \underline{E}_u \cong E$ and $\text{Gal}(\underline{E}'/\underline{E}) \cong \text{Gal}(E'/E)$, and both w and u are inert in \underline{E}' .

(II) Let $\varphi^* : G \rightarrow G^*$ be the inner twisting such that G^* is quasi-split. Since G, H and T split over E' , the homomorphisms ρ_G, ρ_H and ρ_T factor through $\text{Gal}(E'/E)$. We denote by \underline{G}^* (resp. \underline{H} , resp. \underline{T}) the quasi-split group over \underline{E} such that $\widehat{\underline{G}^*} = \widehat{G}$ (resp. $\widehat{\underline{H}} = \widehat{H}$, resp. $\widehat{\underline{T}} = \widehat{T}$) and $\rho_{\underline{G}^*}$ (resp. $\rho_{\underline{H}}$, resp. $\rho_{\underline{T}}$) is the composition of the projection $\underline{\Gamma} \rightarrow \text{Gal}(\underline{E}'/\underline{E}) \cong \text{Gal}(E'/E)$ and the homomorphism $\rho_G : \text{Gal}(E'/E) \rightarrow \text{Out}(\widehat{G})$ (resp. $\rho_H : \text{Gal}(E'/E) \rightarrow \text{Out}(\widehat{H})$, resp. $\rho_T : \text{Gal}(E'/E) \rightarrow \text{Out}(\widehat{T})$).

By construction, we have $\underline{G}_w^* \cong G^*$, $\underline{H}_w \cong H$ and $\underline{T}_w \cong T$. Moreover, the conjugacy classes of embeddings $\widehat{T} \hookrightarrow \widehat{G}$ and $\widehat{T} \hookrightarrow \widehat{H}$ corresponding to $a : T \hookrightarrow G$ and $[b]$ are $\underline{\Gamma}$ -invariant. Therefore they come from stable conjugacy classes of embeddings $\underline{a}^* : \underline{T} \hookrightarrow \underline{G}^*$ and $\underline{b} : \underline{T} \hookrightarrow \underline{H}$. Furthermore, $(\underline{a}^*)_w$ is stably conjugate to a , and $[\underline{b}]_w = [b]$.

(III) Since u is inert in $\underline{E}' = \underline{E}[\underline{T}] \supset \underline{E}[\underline{G}^*]$, the canonical map $H^1(\underline{E}, (\underline{G}^*)^{\text{ad}}) \rightarrow \bigoplus_{v \neq u} H^1(\underline{E}_v, (\underline{G}^*)^{\text{ad}})$ is an isomorphism (by Lemma B.3.2 (b)). Hence there exist unique inner twistings $\varphi^* : \underline{G}^* \rightarrow \underline{G}$ and $\varphi : \underline{G} \rightarrow \underline{G}'$ such that $\varphi_w^* \cong (\varphi^*)^{-1}$, $\varphi_w \cong \varphi$, while φ_v^* and φ_v are trivial for all $v \neq w, u$.

Applying Lemma B.3.2 (c) for the embedding $\underline{a}^* : \underline{T}/Z(\underline{G}) \hookrightarrow (\underline{G}^*)^{\text{ad}}$ we conclude from 1.1.4 (b) that there exist embeddings $\underline{a} : \underline{T} \hookrightarrow \underline{G}$ and $\underline{a}' : \underline{T} \hookrightarrow \underline{G}'$ stably conjugate to \underline{a}^* . Applying now Lemma B.3.2 (c) for \underline{T} , we can further replace \underline{a} and

\underline{a}' so that \underline{a}_w is conjugate to a , \underline{a}'_w is conjugate to a' , while \underline{a}_v and \underline{a}'_v are conjugate for all $v \neq w, u$.

(IV) Since w is inert in $\underline{E}[T] \supset \underline{E}[H]$, we get that $Z(\widehat{H})^\Gamma = Z(\widehat{H})^\Gamma$, and the conjugacy class $[\eta]$ of embeddings $\widehat{H} = \widehat{H} \hookrightarrow \widehat{G} = \widehat{G}$ is Γ -invariant. Hence the triple $\underline{\mathcal{E}} := (\underline{H}, [\eta], \bar{s})$ is an endoscopic triple for \underline{G} . Moreover, $\underline{\mathcal{E}}_w \cong \mathcal{E}$ and $\underline{\mathcal{E}}_u$ is elliptic. Indeed, u is inert in $\underline{E}[H]$, therefore $Z(\widehat{H})^{\Gamma_u} = Z(\widehat{H})^\Gamma = Z(\widehat{H})^\Gamma$ is finite. \square

B.4. Proof of Theorem B.1.4.

B.4.1. Fix $(a, a'; [b])$ -indistinguishable $\phi \in \mathcal{S}(\mathcal{G}(E))$ and $\phi' \in \mathcal{S}(\mathcal{G}'(E))$. We want to show that $e'(G)\mathcal{F}(\phi)$ and $e'(G')\mathcal{F}(\phi')$ are $(a, a'; [b])$ -indistinguishable. Using Lemma 1.6.11 and the symmetry between G and G' , it will suffice to check that for each compatible $x \in \mathcal{G}^{\text{sr}}(E)$ and $y \in \mathcal{H}^{\text{sr}}(E)$ we have:

- (i) $O_x^{[y]}(\mathcal{F}(\phi)) = 0$, if x does not have a stable conjugate in $\mathcal{G}'(E)$;
- (ii) $e'(G)O_x^{[y]}(\mathcal{F}(\phi)) = e'(G') \left\langle \frac{x, x'; [y]}{a, a'; [b]} \right\rangle O_{x'}^{[y]}(\mathcal{F}(\phi'))$, for each stable conjugate $x' \in \mathcal{G}'^{\text{sr}}(E)$ of x .

B.4.2. Fix x and y as in B.4.1. Let $\Omega_x \subset \mathcal{G}_x^{\text{sr}}(E)$ be an open neighborhood of x such that $O_{\tilde{x}}^{\bar{\kappa}[y]}(\mathcal{F}(\phi)) = O_x^{\bar{\kappa}[y]}(\mathcal{F}(\phi))$ for each $\tilde{x} \in \Omega_x$, and $O_{\tilde{x}'}^{\bar{\kappa}[y]}(\mathcal{F}(\phi')) = O_{x'}^{\bar{\kappa}[y]}(\mathcal{F}(\phi'))$ for each stable conjugate $x' \in \mathcal{G}'(E)$ of x and each $\tilde{x}' \in \varphi_{x, x'}(\Omega_x) \subset \mathcal{G}'^{\text{sr}}(E)$. Denote by $\Omega_y \subset \mathcal{H}_y^{\text{sr}}(E)$ the image of Ω_x under the natural isomorphism $\mathcal{G}_x \xrightarrow{\sim} \mathcal{H}_y$, sending x to y , and choose an open neighborhood $\Omega \subset \mathcal{H}^{\text{sr}}(E)$ of y contained in $\text{Ad } H(E)(\Omega_y)$.

By construction, for each $\tilde{y} \in \Omega$ there exists $\tilde{x} \in \Omega_x \subset \mathcal{G}^{\text{sr}}(E)$ compatible with y . Moreover, since conditions (i), (ii) from B.4.1 do not change when y and x are replaced by stable conjugates, it will suffice to show (i), (ii) for some pair of compatible elements $\tilde{y} \in \Omega \subset \mathcal{H}^{\text{sr}}(E)$ and $\tilde{x} \in \mathcal{G}^{\text{sr}}(E)$.

B.4.3. Strategy of the proof.

Choose \underline{E} , u , w , \underline{G} , $\underline{\varphi}$, $\underline{\mathcal{E}}$, \underline{T} , \underline{a} , \underline{a}' , $[\underline{b}]$ as in Claim B.3.6, and identify \underline{G}_v with \underline{G}'_v for each $v \neq w, u$ as in Claim B.3.6 (b). Finally, we fix a non-trivial character $\psi : \mathbb{A}_F/F \rightarrow \mathbb{C}^\times$.

Our strategy will be to construct measures $\underline{\phi} = \otimes_{v \in V} \phi_v \in \mathcal{S}(\mathcal{G}(\mathbb{A}))$ and $\underline{\phi}' = \otimes_{v \in V} \phi'_v \in \mathcal{S}(\mathcal{G}'(\mathbb{A}))$ and compatible elements $\underline{y} \in \underline{\mathcal{H}}^{\text{sr}}(\underline{E})$ and $\underline{x} \in \underline{\mathcal{G}}^{\text{sr}}(\underline{E})$ satisfying the following properties:

- (A) $\underline{y}_w \in \Omega$, $\phi_w = \phi$ and $\phi'_w = \phi'$.
- (B) \underline{x} has a stable conjugate in $\underline{\mathcal{G}}'(\underline{E})$ if \underline{x}_w has a stable conjugate in $\mathcal{G}'(E)$.
- (C) both $\underline{\phi}$ and $\underline{\phi}'$ satisfy the support assumption of Proposition B.3.5 (a), and we have $\Theta(\underline{\phi}) = \Theta(\underline{\phi}')$.

(D) $\mathcal{F}(\underline{\phi})$ and $\mathcal{F}(\underline{\phi}')$ satisfy the support assumption of Proposition B.3.5 (a), and we have $\Theta(\mathcal{F}(\underline{\phi})) = O_{\underline{x}}^{[y]}(\mathcal{F}(\underline{\phi}))$ and $\Theta(\mathcal{F}(\underline{\phi}')) = O_{\underline{x}}^{[y]}(\mathcal{F}(\underline{\phi}'))$. (We define $O_{\underline{x}}^{[y]}(\mathcal{F}(\underline{\phi}'))$ to be zero unless there exists a stable conjugate $\underline{x}' \in \underline{\mathcal{G}}'(\underline{E})$ of \underline{x} , in which case we define $O_{\underline{x}}^{[y]}(\mathcal{F}(\underline{\phi}'))$ to be $O_{\underline{x}'}^{[y]}(\mathcal{F}(\underline{\phi}'))$.)

(E) For each $v \neq w$, we have $O_{\underline{x}_v}^{[y]^v}(\mathcal{F}(\phi_v)) \neq 0$.

(F) For each $v \in V_f \setminus w$, there exists a stable conjugate $x'_v \in \underline{\mathcal{G}}'(\underline{E}_v)$ of \underline{x}_v , and we have

$$e'(\underline{G}_v) O_{\underline{x}_v}^{[y]^v}(\mathcal{F}(\phi_v)) = e'(\underline{G}'_v) \left\langle \frac{x_v, x'_v; [y]_v}{a_v, a'_v; [b]_v} \right\rangle_{\underline{\mathcal{E}}_v} O_{x'_v}^{[y]^v}(\mathcal{F}(\phi'_v)).$$

Once these data are constructed, the result follows. Indeed, by (A) and the observation at the end of B.4.2, it will suffice to check that \underline{y}_w and \underline{x}_w satisfy conditions (i), (ii) of B.4.1. Next (C), (D) and Proposition B.3.5 (b) imply that $O_{\underline{x}}^{[y]}(\mathcal{F}(\underline{\phi})) = O_{\underline{x}}^{[y]}(\mathcal{F}(\underline{\phi}'))$. Assume first that \underline{x}_w does not have a stable conjugate in $\mathcal{G}'(E)$. Then \underline{x} does not have a stable conjugate in $\underline{\mathcal{G}}'(\underline{E})$, thus $O_{\underline{x}}^{[y]}(\mathcal{F}(\underline{\phi})) = \prod_v O_{\underline{x}_v}^{[y]^v}(\mathcal{F}(\phi_v)) = 0$. Hence the vanishing of $O_{\underline{x}_w}^{[y]^w}(\mathcal{F}(\phi_w))$ follows from (E).

Assume now that \underline{x}_w has a stable conjugate in $\mathcal{G}'(E)$, then by (B), there exists a stable conjugate $\underline{x}' \in \underline{\mathcal{G}}'(\underline{E})$ of \underline{x} . Then $O_{\underline{x}}^{[y]}(\mathcal{F}(\underline{\phi})) = O_{\underline{x}'}^{[y]}(\mathcal{F}(\underline{\phi}'))$ and each $\underline{x}'_v \in \underline{\mathcal{G}}'(\underline{E}_v)$ is a stable conjugate of \underline{x}_v . Using product formulas $\prod_{v \in V} e'(\underline{G}_v) = \prod_{v \in V} e'(\underline{G}'_v) = 1$ (see [Ko5]) and $\prod_{v \in V_f} \left\langle \frac{x_v, x'_v; [y]_v}{a_v, a'_v; [b]_v} \right\rangle_{\underline{\mathcal{E}}_v} = 1$ (B.3.3 (c)), the required equality $e'(G) O_{\underline{x}_w}^{[y]^w}(\mathcal{F}(\phi)) = e'(G') \left\langle \frac{x_w, x'_w; [y]_w}{a_w, a'_w; [b]_w} \right\rangle O_{x'_w}^{[y]^w}(\mathcal{F}(\phi'))$ follows from (E) and (F).

B.4.4. Construction of $\underline{\phi}, \underline{\phi}', \underline{y}$ and \underline{x} .

(a) Choose an $\underline{\mathcal{Q}}$ -subalgebra $\mathcal{K} \subset \underline{\mathcal{G}}(\underline{E})$, and let $S_1 \subset V$ be a finite subset containing $V_\infty \cup \{w, u\}$ such that for each $v \notin S_1$ we have

- \underline{H}_v and $\underline{\mathcal{G}}_v$ are unramified;
- the $\underline{\mathcal{Q}}_v$ -subalgebra $\mathcal{K}_v \subset \underline{\mathcal{G}}(\underline{E}_v)$, spanned by \mathcal{K} , is hyperspecial and satisfies $\mathcal{F}(1_{\mathcal{K}_v}) = 1_{\mathcal{K}_v}$.

(b) Let \mathcal{A} be the set of isomorphism classes of those endoscopic triples for \underline{G} , which are unramified outside of S_1 . Then \mathcal{A} is finite (see [La1, Lem. 8.12]), and $\underline{\mathcal{E}} \in \mathcal{A}$. Let \mathcal{A}' be the subset of \mathcal{A} consisting of triples $(H_a, [\eta_a], \overline{s}_a)$ such that $\underline{E}[H_a]$ is not contained in $\underline{E}[\underline{H}]$. For each $a \in \mathcal{A}'$, we fix a prime $v_a \in V_f \setminus S_1$ which splits in $\underline{E}[\underline{H}]$ but does not split in $\underline{E}[H_a]$. Put $S_2 := \{v_a \mid a \in \mathcal{A}'\}$, and set $S := S_1 \cup S_2$.

(c) Choose $\underline{y} \in \mathcal{H}^{\text{sr}}(\underline{E})$ such that $\underline{y}_w \in \Omega$, $\underline{y}_u \in \mathcal{H}^{\text{sr}}(\underline{E}_u)$ is elliptic (that is, $\underline{H}_{\underline{y}_u} \subset \underline{H}_u$ is elliptic) and $\underline{y}_v \in \mathcal{H}^{\text{sr}}(\underline{E}_v)$ is split (that is, $\underline{H}_{\underline{y}_v} \subset \underline{H}_v$ is split) for each $v \in S_2$.

Choose an element $\underline{x}^* \in \underline{\mathcal{G}}^*(\underline{E})$ compatible with \underline{y} (exists by Lemma 1.1.10 (b)). Then $(\underline{G}_{\underline{x}^*}^*)_u \subset \underline{G}_u^*$ is an elliptic torus. Since $\underline{y}_w \in \Omega$, element \underline{x}_w^* has a stable conjugate in $\Omega_x \subset \underline{\mathcal{G}}_x^{\text{sr}}(\underline{E})$. Since φ_v^* is trivial for each $v \neq w, u$, it follows from Lemma B.3.2 (c) (ii) (as in the proof Claim B.3.6 (III)) that there exists a stably conjugate $\underline{x} \in \underline{\mathcal{G}}(\underline{E})$ of \underline{x}^* .

(d) Choose a stably conjugate $x'_u \in \underline{\mathcal{G}}'(\underline{E}_u)$ of \underline{x}_u (which exists by Lemma 1.5.3), and set $x'_v := \underline{x}_v$ for each $v \neq w, u$. For each $v \in S \setminus (V_\infty \cup w)$, choose measures $\phi_v \in \mathcal{S}(\underline{G}(\underline{E}_v))$ and $\phi'_v \in \mathcal{S}(\underline{G}'(\underline{E}_v))$ constructed in Corollary B.2.6 for the triple $(\underline{y}_v, \underline{x}_v, x'_v)$. In particular, $\phi_v = \phi'_v$ for each $v \in S \setminus (V_\infty \cup \{u, w\})$.

For each $v \in S \setminus V_\infty$, let $\omega_v \subset \underline{E}_v^\times$ be an open neighborhood of the identity such that Ω is invariant under the multiplication by ω_w , while $\mathcal{F}(\phi_v)$ and $\mathcal{F}(\phi'_v)$ are invariant under the multiplication by ω_v if $v \neq w$.

For each $v \in V \setminus S$, put $\phi_v = \phi'_v = 1_{\mathcal{K}_v} dx_v$. Finally, put $\phi_w := \phi$ and $\phi'_w := \phi'$.

(e) Choose a finite set $S_3 \supset S$ such that for each $v \notin S_3$, we have $\underline{x}_v \in \mathcal{K}_v$, and the reduction of \underline{x}_v modulo v is regular. Choose $\lambda \in \underline{E}^\times$ such that

- (i) $\text{val}_v(\lambda) = 0$ for each $v \notin S_3$;
- (ii) $\lambda \in \omega_v$ for each $v \in S \setminus V_\infty$;
- (iii) For each $v \in S_3 \setminus S$, $\text{val}_v(\lambda)$ is so large that $(\lambda \bar{y})_v$ belongs to the open neighborhood of zero prescribed in Lemma B.2.7 (b).

Finally, we replace \underline{y} and \underline{x} constructed in (c) by $\lambda \underline{y}$ and $\lambda \underline{x}$, respectively.

(f) Recall that $\mathcal{F}(\phi_v)$ and $\mathcal{F}(\phi'_v)$ are compactly supported for each $v \in V_f$ and $\mathcal{F}(\phi_v) = \mathcal{F}(\phi'_v) = 1_{\mathcal{K}_v} dx_v$ for each $v \notin S$. Since $\underline{E} \subset \mathbb{A}$ is discrete, one can choose a compact neighborhood $C_v \subset \underline{\mathcal{G}}(\underline{E}_v) = \underline{\mathcal{G}}'(\underline{E}_v)$ of \underline{x}_v for each $v \in V_\infty$ such that all elements of $\underline{\mathcal{G}}(\underline{E}) \cap (\prod_{v \in V_\infty} C_v \times \prod_{v \in V_f} \text{Supp}(\mathcal{F}(\phi_v)))$ and $\underline{\mathcal{G}}'(\underline{E}) \cap (\prod_{v \in V_\infty} C_v \times \prod_{v \in V_f} \text{Supp}(\mathcal{F}(\phi'_v)))$ are stable conjugate to \underline{x} .

For each $v \in V_\infty$, choose a measure $\phi_v = \phi'_v$ of $\mathcal{S}(\underline{\mathcal{G}}(\underline{E}_v))$ such that $\mathcal{F}(\phi_v) = f_v dx_v$ for a smooth non-negative function f_v on $\underline{\mathcal{G}}(\underline{E}_v)$ supported on C_v such that $f_v(\underline{x}_v) \neq 0$. Put $\underline{\phi} := \otimes_{v \in V} \phi_v$ and $\underline{\phi}' := \otimes_{v \in V} \phi'_v$.

To complete the proof of Theorem B.1.4, it remains to show that the constructed above $\underline{\phi}, \underline{\phi}', \underline{y}$ and \underline{x} satisfy conditions (A) – (F) of B.4.3.

B.4.5. Proof of conditions (A) – (F).

(A) is clear (see B.4.4 (c) and (e)).

(B) Since \underline{x}_u is elliptic, the assertion follows from Lemma B.3.2 (c) (ii) (as in the proof of Claim B.3.6 (III)).

(C) Since the support of ϕ_u and ϕ'_u is regular elliptic, both $\underline{\phi}$ and $\underline{\phi}'$ satisfy the support assumption of Proposition B.3.5 (a). Because of symmetry between $\underline{\phi}$ and $\underline{\phi}'$, it will therefore suffice to check that for every $\underline{z} \in \underline{\mathcal{G}}(\underline{E})$ and $\underline{\kappa} \in \widehat{\underline{G}}_{\underline{z}}^{\Gamma}$ such that $O_{\underline{z}}^{\underline{\kappa}}(\underline{\phi}) \neq 0$, there exists a stable conjugate $\underline{z}' \in \underline{\mathcal{G}}'(\underline{E})$ of \underline{z} , and we have $O_{\underline{z}}^{\underline{\kappa}}(\underline{\phi}) = O_{\underline{z}'}^{\underline{\kappa}}(\underline{\phi}')$.

Fix $\underline{z} \in \underline{\mathcal{G}}(\underline{E})$ and $\underline{\kappa} \in \widehat{\underline{G}}_{\underline{z}}^{\Gamma}$ such that $O_{\underline{z}}^{\underline{\kappa}}(\underline{\phi}) \neq 0$. Consider the endoscopic triple $\mathcal{E}' := \mathcal{E}_{([a_{\underline{z}}], \underline{\kappa})} = (H', [\eta'], \overline{s}')$ for \underline{G} . Following [Wa1, 10.9], we will show that $\mathcal{E}' \cong \underline{\mathcal{E}}$.

By the assumption, $O_{\underline{z}_v}^{\underline{\kappa}_v}(\phi_v) \neq 0$ for each $v \in V$. Since $\mathcal{K}_v \subset \underline{\mathcal{G}}(\underline{E}_v)$ is a hyper-special subalgebra and $\phi_v = 1_{\mathcal{K}_v} dx_v$ for each $v \in V \setminus S$, we conclude from Lemma B.2.7 (a) that \mathcal{E}' is unramified outside of S .

For each $v \in S_2$, measure ϕ_v is supported on split elements (by Corollary B.2.6 (i)) and satisfies $O_{\underline{z}_v}^{\underline{\kappa}_v}(\phi_v) \neq 0$. Therefore $\underline{G}_{\underline{z}_v}$ is split, hence $\mathcal{E}'_v \cong \mathcal{E}_{([a_{\underline{z}_v}], \underline{\kappa}_v)}$ is split. In particular, \mathcal{E}' is unramified outside of S_1 . Moreover, H' splits at each $v \in S_2$, hence $\underline{E}[H'] \subset \underline{E}[H]$.

Since $O_{\underline{z}_u}^{\underline{\kappa}_u}(\phi_u) \neq 0$, we get by Corollary B.2.6 (i), (iii) that \underline{z}_u is stable conjugate to an element of $\underline{G}_{\underline{x}_u}^{\text{sr}}(\underline{E}_u)$, and the class of $\underline{\kappa}_u \in \widehat{\underline{G}}_{\underline{z}}^{\Gamma_u} \cong \widehat{\underline{G}}_{\underline{x}}^{\Gamma_u}$ equals $\underline{\kappa}_{[y]_u}$. Since \underline{x}_u is elliptic, we get that $\mathcal{E}'_u \cong \mathcal{E}_{([a_{\underline{x}_u}], \underline{\kappa}_{[y]_u})}$ is isomorphic to $\underline{\mathcal{E}}_u$ (use Lemma 1.3.9 (a)). As u is inert in $\underline{E}[H] \supset \underline{E}[H']$, we conclude from this that $\mathcal{E}' \cong \underline{\mathcal{E}}$ (use B.3.3 (b)), as claimed.

By the proven above, there exists $\underline{y}' \in \underline{\mathcal{H}}^{\text{sr}}(\underline{E})$ compatible with \underline{z} such that $\underline{\kappa} = \underline{\kappa}_{[y']}$, thus $O_{\underline{z}}^{\underline{\kappa}}(\underline{\phi}) = O_{\underline{z}}^{[y']}(\underline{\phi}) \neq 0$. In particular, $O_{\underline{z}_w}^{[y']_w}(\phi_w) \neq 0$. Since ϕ_w and ϕ'_w are $(a, a'; [b])$ -indistinguishable, it follows from Lemma 1.6.11 that there exists a stable conjugate $\underline{z}'_w \in \underline{\mathcal{G}}'(\underline{E})$ of \underline{z}_w . Since $\underline{G}_{\underline{z}_u} \subset \underline{G}_u$ is elliptic, there exists $\underline{z}' \in \underline{\mathcal{G}}'(\underline{E})$ stably conjugate to \underline{z} such that \underline{z}'_v is conjugate to \underline{z}_v for each $v \neq u, w$ (use Lemma B.3.2 (c) and 1.1.4).

It now remains to show that $O_{\underline{z}}^{[y']}(\underline{\phi}) = O_{\underline{z}'}^{[y']}(\underline{\phi}')$. By the product formula (B.3.3 (c)), it will suffice to check that for each $v \in V_f$, we have

$$O_{\underline{z}_v}^{[y']_v}(\phi_v) = \left\langle \frac{\underline{z}_v, \underline{z}'_v; \underline{y}'_v}{\underline{a}_v, \underline{a}'_v; [\underline{b}]_v} \right\rangle_{\underline{\mathcal{E}}_v} O_{\underline{z}'_v}^{[y']_v}(\phi'_v).$$

The assertion for $v = w$ follows from Lemma 1.6.11, while the assertion for $v = u$ follows from Corollary B.2.6 (ii). Finally, the assertion for $v \neq u, w$ follows from the fact that under the identification $\underline{G}'_v = \underline{G}_v$, we have $\phi'_v = \phi_v$, \underline{z}'_v is conjugate to \underline{z}_v , and \underline{a}'_v is conjugate to \underline{a}_v .

(D) By B.4.4 (f), $\text{Supp}(\mathcal{F}(\underline{\phi})) \cap \text{Ad } \underline{G}(\mathbb{A})(\mathcal{G}(\underline{E}))$ consists of elements stably conjugate to \underline{x} . Since \underline{x}_u is elliptic, \underline{x} is elliptic, thus $\mathcal{F}(\underline{\phi})$ satisfies the support assumption

of Proposition B.3.5 (a). Therefore $\Theta(\mathcal{F}(\underline{\phi})) = \sum_{\underline{\kappa} \in \widehat{G}_{\underline{x}}^{\Gamma}} O_{\underline{x}}^{\underline{\kappa}}(\mathcal{F}(\underline{\phi}))$. Let $\underline{\kappa} \in \widehat{G}_{\underline{x}}^{\Gamma}$ be such that $O_{\underline{x}}^{\underline{\kappa}}(\mathcal{F}(\underline{\phi})) \neq 0$. Then $O_{\underline{x}_u}^{\underline{\kappa}_u}(\mathcal{F}(\phi_u)) \neq 0$. Since \underline{x}_u is elliptic, it follows from Corollary B.2.6 (iv) that $\underline{\kappa}_u = \overline{\kappa}_{[y]_u}$. But the map $\widehat{G}_{\underline{x}}^{\Gamma} \hookrightarrow \widehat{G}_{\underline{x}}^{\Gamma_u} = \pi_0(\widehat{G}_{\underline{x}}^{\Gamma_u})$ is injective, therefore $\underline{\kappa} = \overline{\kappa}_{[y]}$. This shows that $\Theta(\mathcal{F}(\underline{\phi})) = O_{\underline{x}}^{[y]}(\mathcal{F}(\underline{\phi}))$. The proof for ϕ' is similar.

(E), (F) For $v \in S \setminus (V_{\infty} \cup w)$, the assertions follows from Corollary B.2.6 (v). For $v \neq u, w$, (F) follows from the fact that under the identification $\underline{G}_v = \underline{G}'_v$, we have $\phi'_v = \phi_v$, and \underline{a}'_v is conjugate to \underline{a}_v . It remains to show (E) for $v \in V_{\infty} \cup (V \setminus S)$. If $v \in V_{\infty}$, the assertion follows from the fact that $\mathcal{F}(\phi_v) = f_v dx_v$, while f_v is non-negative and satisfies $f_v(\underline{x}_v) \neq 0$. Assume now that $v \notin S$. Then $\mathcal{F}(\phi_v) = 1_{\kappa_v} dx_v$. The assertion now follows from Lemma B.2.7 (c) if $v \notin S_3$ and from the choice of λ in B.4.4 (e) (and Lemma B.2.7 (b)) if $v \in S_3$.

LIST OF MAIN TERMS AND SYMBOLS

\mathcal{E} -admissible, 26	(G, a_0, γ_0) -relevant, 61	$\mathcal{D}^0(X(E))$, 28
$(\mathcal{E}, [a], \overline{\kappa})$ -admissible, 26	\mathcal{E} -stable, 28	$dg = \omega_G $, 27
compact element, 50	stably conjugate, 6	$dx = \omega_X $, 27
compatible, 15, 71	strongly regular, 8	\overline{E} , 4
\mathcal{E} -compatible, 13	a -strongly regular, 43	E^{sep} , 4
conjugate, 6	topological Jordan decomposition, 51	E^{nr} , 4
E^{sep} -conjugate, 6	topologically unipotent, 51	$e(G)$, $e'(G)$, 4
defined at x , 38	quasi-isogeny, 9	$\mathcal{E} = (H, [\eta], \overline{s})$, 13
defined over \mathcal{O} , 39	quasi-logarithm, 36	$\mathcal{E}_{([a], \kappa)}$, 15
Deligne–Lusztig representation, 43	\mathcal{E} -unstable, 28	$\mathcal{E}_t = (H_t, [\eta_t], \overline{s}_t)$, 61
endoscopic triple, 13	$a : T \hookrightarrow G$, 43	$F _U$, 27
$(a, a'; [b])$ -equivalent, 29	$\overline{a} : \overline{T} \hookrightarrow L_a$, 43	$F(x, \overline{\xi})$, 31
$(a, a'; \kappa)$ -equivalent, 30	$\text{Ad } G$, $\text{Ad } g$, 4	$F_{a, \theta}$, 44
$(\gamma_0, \overline{\xi})$ -equivalent, 63, 65	$\widehat{[a]}$, 10	$F_{a_0, \kappa, \theta}$, 44
Fourier transform, 48	a_x , 8	F_{tu} , 45
$(a, a'; [b])$ -indistinguishable, 29	\overline{a} , 38	F_x, F_{x^*} , 35
inner twisting, 6	$\mathcal{B}(G)$, 37	\mathcal{F} , 48
non-degenerate at x , 38	$[b]_G$, 11	G^0 , 4
non-degenerate over \mathcal{O} , 39	$b_{t, \delta}$, 61	G^{ad} , G^{der} , 4
property (vg) , 39	$C_c^{\infty}(X(E))$, 27	G^{sc} , 4
property $(vg)_a$, 43	$\mathcal{D}(X(E))$, 27	\mathcal{G} , 4
	\mathcal{D}_G , 23	$G(E)_{\text{tu}}$, $\mathcal{G}(E)_{\text{tn}}$, 40
		\mathbb{F}_q , 4

0G , 41	$\mathcal{S}(X(E))$, 27	$\pi_{a,\theta}$, 43
G^* , 27	\mathcal{T} , 4	$[\widehat{\pi}]$, 11
\widehat{G} , 10	\underline{Tor} , 32	$\rho_{a,\theta}$, 43
G_a, \mathcal{G}_a , 43	$T_{\mathcal{O}}, \overline{T}$, 38	$\rho_{\overline{a},\overline{\theta}}$, 43
$G_{a^+}, \mathcal{G}_{a^+}$, 43	$t_{a,\theta}$, 44	$\rho_G : \Gamma \rightarrow \text{Out}(\widehat{G})$, 10
\widetilde{G}_a , 43	$\mathcal{U}(H)$, 40	$\Phi : G \rightarrow \mathcal{G}$, 36
G_x, \mathcal{G}_x , 4, 37	$W(G)$, 4	$\Phi_x : \underline{G}_x \rightarrow \mathcal{G}_x$, 38
$G_{x^+}, \mathcal{G}_{x^+}$, 37	X^{sr} , 8	$\overline{\Phi}_x : L_x \rightarrow \mathcal{L}_x$, 38
$G_{x,\text{tu}}, \mathcal{G}_{x,\text{tn}}$, 40	X^* , 27	${}^0\Phi$, 46
$\underline{G}_x, \overline{G}_x$, 37	$[x]$, 8	Φ_ρ , 36
\overline{g} , 38	$[y]_G$, 70	$\phi _U$, 27
\mathcal{H} , 4	$Z(G)$, 4	$\varphi : G \rightarrow G'$, 6
\mathcal{I}^+ , 48	$Z(\mathcal{E})$, 15	$\varphi_X : X \rightarrow X'$, 6
$\text{Int } G, \text{Int } g$, 4	$Z(\mathcal{E}, [a], \overline{\kappa})$, 15	$\varphi_{x,x'}$, 33
$\text{inv}(G, G')$, 6	$Z[\widehat{a}]$, 11	$\chi(\pi_{a,\theta})$, 44
$\text{inv}(x, x')$, 6	$Z[\overline{\eta}]$, 11	$\chi_{a_0,\kappa,\theta}$, 44
$\overline{\text{inv}}(x, x')$, 6	Γ , 4	$\Omega_{a,t}, \overline{\Omega}_{a,t}$, 46
$\overline{\text{inv}}((x_1, x'_1); \dots; (x_k, x'_k))$, 7	$\Delta_{a,t}$, 47	$\omega_G, \omega_{\mathcal{G}}$, 27
\mathcal{L} , 4	$\Delta_{a_0,\kappa,t}$, 47	ω_X , 27
$L_{\text{loc}}^1(X(E))$, 31	$\delta_1 \sim_{(\gamma_0, \overline{\xi})} \delta_2$, 63, 66	$\omega_{X'}$, 27
L_a, \mathcal{L}_a , 43	$\delta_{a,t}, \overline{\delta}_{a,t}$, 46	$\left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle$, 24
L_x, \mathcal{L}_x , 38	$\theta : T(E) \rightarrow \mathbb{C}^\times$, 43	$\left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; [b_2]} \right\rangle_{\mathcal{E}}$, 24
\mathfrak{m} , 4	$\overline{\theta} : \overline{T}(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$, 43	$\left\langle \frac{a_1, a'_1; \overline{\kappa}}{a_2, a'_2; [b_2]} \right\rangle$, 26
$\mathcal{N}(\mathcal{H})$, 40	$\iota([b_1], [b_2])$, 12	$\left\langle \frac{a_1, a'_1; [b_1]}{a_2, a'_2; \overline{\kappa}} \right\rangle$, 26
\mathcal{O} , 4	$\kappa([b_1], [b_2])$, 12	$\left\langle \frac{x, x'; \overline{\kappa}}{a, a'; [b]} \right\rangle$, 26
O_x , 28	$\kappa\left(\frac{[b_1]}{[b_2]}\right)$, 13	$\left\langle \frac{x, x'; [y]}{a, a'; [b]} \right\rangle$, 71
O_x^ξ , 28	$\overline{\kappa}_{[b]}$, 15	$\langle \cdot, \cdot \rangle_x$, 38
$O_x^{[y]}$, 71	$\overline{\kappa}_{[y]}$, 71	$\langle \cdot, \cdot \rangle_\rho$, 36
p , 4	$\Lambda(\mathcal{E})$, 14	
SO_x, SO_{x^*} , 28	μ_H, ν_H , 12	
$S_{([a], \overline{\kappa})}$, 15	$\Pi_{\mathcal{E}}$, 15	
$S_{[b]}$, 18	$\pi_{\mathcal{E}}$, 14	

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